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THE STABLE DERIVATION ALGEBRAS FOR HIGHER GENERA

BY

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ABSTRACT

In this article, we consider certain systems of derivation algebras related to Galois representations attached to fundamental groups of algebraic curves of positive genera and establish some stability property. This is a generalization of Ihara's result in the case of genus zero.

0. Introduction

Let C be a non-singular curve of genus $g \geq 1$ with one point punctured over the rationals $\mathbf{Q}$. For a fixed prime number l, we have a Galois representation

$$(0.0.1) \quad \varphi_C^{\text{pro}-l}\colon \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Out}\,\pi_1^{\text{pro}-l}(\bar{C}) = \mathrm{Aut}\,\pi_1^{\text{pro}-l}(\bar{C})/\,\mathrm{Int}\,\pi_1^{\text{pro}-l}(\bar{C})$$

attached to the pro-l fundamental group of $\bar{C} = C \times_{\mathbf{Q}} \bar{\mathbf{Q}}$. Our primary subject is to determine the image of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ under $\varphi_C^{\text{pro}-l}$. For this purpose it is useful to consider configuration spaces of points on C and the system of Galois representations attached to their pro-l fundamental groups. Let $C^{(r)}$ be the configuration space of ordered r points on C:

$$(0.0.2) \quad \begin{aligned} C^{(r)} &= \underbrace{C \times \cdots \times C}_{r} \setminus (\text{weak diagonal}) \\ &= \{(P_1, \ldots, P_r) \mid P_i \in C, P_i \neq P_j (i \neq j)\}, \end{aligned}$$

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and consider a sequence of projections

$$(0.0.3)\qquad \cdots \longrightarrow C^{(r)} \xrightarrow{p_r^{(r)}} C^{(r-1)} \longrightarrow \cdots \longrightarrow C^{(2)} \xrightarrow{p_2^{(2)}} C^{(1)} = C,$$

where $p_r^{(r)}\colon C^{(r)} \to C^{(r-1)}$ is defined by forgetting the r-th point. We consider a system of Galois representations

$$(0.0.4)\qquad \varphi_{C^{(r)}}^{\mathrm{pro}-l}\colon \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Out}\,\pi_1^{\mathrm{pro}-l}(\bar{C}^{(r)})$$

which is compatible with (0.0.3). Speaking a little more precisely, we introduce certain subgroups $\mathrm{Out}^\flat\,\pi_1^{\mathrm{pro}-l}(\bar{C}^{(r)})$ of $\mathrm{Out}\,\pi_1^{\mathrm{pro}-l}(\bar{C}^{(r)})$ which contain the image of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ and which admit homomorphisms $\psi_r^{(r)}\colon \mathrm{Out}^\flat\,\pi_1^{\mathrm{pro}-l}(\bar{C}^{(r)}) \to \mathrm{Out}^\flat\,\pi_1^{\mathrm{pro}-l}(\bar{C}^{(r-1)})$ induced from $p_r^{(r)}$. Then the Galois representations are compatible with these homomorphisms: $\psi_r^{(r)} \circ \varphi_{C^{(r)}}^{\mathrm{pro}-l} = \varphi_{C^{(r-1)}}^{\mathrm{pro}-l}$. Hence the image of $\varphi_C^{\mathrm{pro}-l}\colon \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Out}\,\pi_1^{\mathrm{pro}-l}(\bar{C})$ is included not only in $\mathrm{Out}^\flat\,\pi_1^{\mathrm{pro}-l}(\bar{C})$ but also in all the images of $\mathrm{Out}^\flat\,\pi_1^{\mathrm{pro}-l}(\bar{C}^{(r)})$.

In this paper we consider graded Lie algebraization of this representation. The study of graded Lie algebraization of this kind of representations was originated by Ihara [I1], and is developped in many following works [AK, K, NTs, NTaU] etc. The fundamental group $\pi_1^{\mathrm{pro}-l}(\bar{C})$ is equipped with a central filtration called weight filtration, which coincides with the descending central sequence in this case. We introduce an induced filtration in $\mathrm{Out}\,\pi_1^{\mathrm{pro}-l}(\bar{C})$ and then in $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ via $\varphi_C^{\mathrm{pro}-l}$. Taking the associated graded structures, we obtain graded Lie algebraization φ_C^{Lie} of $\varphi_C^{\mathrm{pro}-l}$:

$$(0.0.5)\qquad \varphi_C^{\mathrm{Lie}}\colon \mathcal{G}_C \longrightarrow \mathrm{Out}^\flat\,\mathrm{Gr}\,\Pi_{g,1}.$$

Moreover, by considering the graded Lie version of $\varphi_{C^{(r)}}^{\mathrm{pro}-l}$, we have a system of homomorphisms of graded Lie algebras

$$(0.0.6)\qquad \varphi_{C^{(r)}}^{\mathrm{Lie}}\colon \mathcal{G}_C \longrightarrow \mathrm{Out}^\flat\,\mathrm{Gr}\,\Pi_{g,1}^{(r)}$$

whose images are preserved in a sequence of graded Lie algebras

$$(0.0.7)\qquad \begin{aligned} &\cdots \longrightarrow \mathrm{Out}^\flat\,\mathrm{Gr}\,\Pi_{g,1}^{(r)} \xrightarrow{\psi_r^{(r)}} \mathrm{Out}^\flat\,\mathrm{Gr}\,\Pi_{g,1}^{(r-1)} \longrightarrow \\ &\qquad \cdots \longrightarrow \mathrm{Out}^\flat\,\mathrm{Gr}\,\Pi_{g,1}^{(2)} \xrightarrow{\psi_2^{(2)}} \mathrm{Out}^\flat\,\mathrm{Gr}\,\Pi_{g,1}^{(1)} = \mathrm{Out}^\flat\,\mathrm{Gr}\,\Pi_{g,1}. \end{aligned}$$

Here, for a Lie algebra $\mathcal{L}$, $\mathrm{Out}\,\mathcal{L}$ means the outer derivation algebra of $\mathcal{L}$, that is, the quotient of the derivation algebra of $\mathcal{L}$ by the Lie ideal consisting of all the inner derivations, and $\mathrm{Out}^\flat$ is defined as a certain subalgebra. Also in this graded Lie case, the Galois image in $\mathrm{Out}^\flat \operatorname{Gr} \Pi_{g,1}$ is contained in all the images of $\mathrm{Out}^\flat \operatorname{Gr} \Pi_{g,1}^{(r)}$ there. Hence our interest goes to the determination of this subalgebra of $\mathrm{Out}^\flat \operatorname{Gr} \Pi_{g,1}$. We call this the **stable derivation algebra** for genus g.

For the sequence (0.0.7), it is known that all homomorphisms $\psi_r^{(r)}\colon \mathrm{Out}^\flat \operatorname{Gr} \Pi_{g,1}^{(r)} \longrightarrow \mathrm{Out}^\flat \operatorname{Gr} \Pi_{g,1}^{(r-1)}$ are injective (Ihara-Kaneko [IK], Nakamura-Takao-Ueno [NTaU]). In this paper we shall prove the surjectivity of these homomorphisms for all r except very small values.

MAIN THEOREM: *For any $g \geq 1$ and $r \geq 4$, the homomorphism*

$$\psi_r^{(r)}\colon \mathrm{Out}^\flat \operatorname{Gr} \Pi_{g,1}^{(r)} \longrightarrow \mathrm{Out}^\flat \operatorname{Gr} \Pi_{g,1}^{(r-1)}$$

is surjective (hence bijective).

The proof is done quite algebraically and it can be formulated as a theorem on graded Lie algebras over $\mathbf{Z}$. We work on a graded Lie algebra $\mathcal{L}_g^{(r)}$ over $\mathbf{Z}$ satisfying that $\mathcal{L}_g^{(r)} \otimes_{\mathbf{Z}} \mathbf{Z}_l \simeq \operatorname{Gr} \Pi_{g,1}^{(r)}$, and a suitable subalgebra $\mathcal{D}_g^{(r)}$ of the derivation algebra $\operatorname{Der} \mathcal{L}_g^{(r)}$ of $\mathcal{L}_g^{(r)}$ satisfying that $\mathcal{D}_g^{(r)} \otimes_{\mathbf{Z}} \mathbf{Z}_l \simeq \mathrm{Out}^\flat \operatorname{Gr} \Pi_{g,1}^{(r)}$. The theorem which shall be actually proved is the following.

MAIN THEOREM: *For any $g \geq 1$ and $r \geq 4$, the homomorphism*

$$\psi_r^{(r)}\colon \mathcal{D}_g^{(r)} \longrightarrow \mathcal{D}_g^{(r-1)}$$

is bijective.

In the case of the projective line with three points punctured, Ihara [I3] defined the stable derivation algebra and proved the stability property. Our main theorem of this paper is an extension of this to curves of positive genera, and we shall use Ihara's result in the most crucial part of the proof of our main theorem.

In Section 1 we give the definition of $\mathcal{L}_g^{(r)}$ and $\mathcal{D}_g^{(r)}$ and state our main theorem precisely. Section 2 is devoted to the proof of the key proposition. For the case of genus one, we try to determine the structure of the stable derivation algebra by the actual computation using computers. We determine the irreducible decomposition of low degree components as $\mathfrak{gl}(2, \mathbf{Q})$-modules. In Section 3 we mention the results of computation. In Section 4 we propose some open problems. In the Appendix we summarize Ihara's result on genus zero case from [I3] by adding some new remarks needed for our discussions in the present paper.

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1. Graded Lie algebras associated with braid groups on a curve of genus g

In this section we shall state basic facts on graded Lie algebras associated with braid groups on a curve of genus g based on [NTaU].

1.1. The graded Lie algebra $\mathcal{L}_g^{(r)}$ over $\mathbf{Z}$ is defined by the following generators and relations:

$$\begin{aligned}
\text{generators: } & X_{ik} = X_{ik}^{(r)}, \quad Y_{ik} = Y_{ik}^{(r)} \quad (1 \leq i \leq g, 1 \leq k \leq r), \\
& Z_{jk} = Z_{jk}^{(r)} \quad (0 \leq j \leq r, 1 \leq k \leq r); \\
\text{grading: } & \deg X_{ik} = \deg Y_{ik} = 1, \deg Z_{jk} = 2; \\
\text{relations: } &
\end{aligned}$$

$$Z_{kk} = 0, \quad Z_{jk} = Z_{kj} \quad (1 \leq j, k \leq r), \tag{1.1.1}$$

$$\sum_{i=1}^{g} [X_{ik}, Y_{ik}] + \sum_{j=0}^{r} Z_{jk} = 0 \quad (1 \leq k \leq r), \tag{1.1.2}$$

$$[Z_{jk}, Z_{j'k'}] = 0 \quad (\{j,k\} \cap \{j',k'\} = \emptyset), \tag{1.1.3}$$

$$[X_{ik}, Z_{jl}] = [Y_{ik}, Z_{jl}] = 0 \quad (k \neq j, l), \tag{1.1.4}$$

$$[X_{ik}, X_{jl}] = [Y_{ik}, Y_{jl}] = 0 \quad (k \neq l), \tag{1.1.5}$$

$$[X_{ik}, Y_{jl}] = 0 \quad (i \neq j, k \neq l), \tag{1.1.6}$$

$$[X_{ik}, Y_{il}] = Z_{kl} \quad (k \neq l). \tag{1.1.7}$$

(We sometimes omit superscripts $^{(r)}$ when no confusion occurs.) When tensored with $\mathbf{Z}_l$, $\mathcal{L}_g^{(r)}$ coincides with $\mathrm{Gr}\,\Pi_{g,1}^{(r)}$ in [NTaU]. The homogeneous component of degree m is denoted by $\mathrm{gr}^m \mathcal{L}_g^{(r)}$. The symmetric group $\mathfrak{S}_r$ acts on $\mathcal{L}_g^{(r)}$ by permutation of indices; $\sigma(X_{ik}) = X_{i\sigma(k)}, \sigma(Y_{ik}) = Y_{i\sigma(k)}, \sigma(Z_{jk}) = Z_{\sigma(j)\sigma(k)}, \sigma(Z_{0k}) = Z_{0\sigma(k)}$.

For $r \geq 2$, $\mathcal{L}_g^{(r)}$ is a successive extension of free Lie algebras as observed later.

We also put

$$\mathcal{L}_{g,<l}^{(r)} = \langle X_{ik}, Y_{ik}, Z_{jk}, Z_{0k} |\, 1 \leq j,k < l \rangle, \tag{1.1.8}$$

$$\mathcal{L}_g^{(r)\circ} = \langle Z_{jk} |\, 0 \leq j \leq r, 1 \leq k \leq r \rangle, \tag{1.1.9}$$

and

$$\mathcal{L}_{g,\neq l}^{(r)\circ} = \langle Z_{jk} |\, j,k \neq l \rangle. \tag{1.1.10}$$

1.2. For $1 \leq k \leq r$, let $\mathcal{N}_k^{(r)}$ be the subalgebra of $\mathcal{L}_g^{(r)}$ generated by X_{ik}, Y_{ik} and Z_{jk} $(1 \leq i \leq g, 0 \leq j \leq r)$. Each of them has one defining relation $\sum_{i=1}^{g}[X_{ik}, Y_{ik}] + \sum_{j=0}^{r} Z_{jk} = 0$ and is free of rank $2g + r$; X_{ik}, Y_{ik} $(1 \leq i \leq g)$ and Z_{jk} $(1 \leq j \leq r)$ form a free generating system. Moreover, $\mathcal{N}_k^{(r)}$ is a Lie ideal of $\mathcal{L}_g^{(r)}$. In fact, $\mathcal{N}_r^{(r)}$ coincides with the kernel of the following "forgetting the r-th string" homomorphism:

$$\begin{aligned} (1.2.1)\quad & p^{(r)} = p_r^{(r)} \colon \mathcal{L}_g^{(r)} \longrightarrow \mathcal{L}_g^{(r-1)}, \\ & X_{ik}^{(r)}, Y_{ik}^{(r)}, Z_{jk}^{(r)} \longmapsto X_{ik}^{(r-1)}, Y_{ik}^{(r-1)}, Z_{jk}^{(r-1)} \quad (1 \leq i \leq g; j,k \neq r), \\ & X_{ir}^{(r)}, Y_{ir}^{(r)}, Z_{jr}^{(r)} \longmapsto 0 \quad (1 \leq i \leq g; 0 \leq j \leq r-1), \end{aligned}$$

and $\mathcal{N}_k^{(r)}$ is the image of $\mathcal{N}_r^{(r)}$ under $\sigma \in \mathfrak{S}_r$ with $\sigma(r) = k$. Hence we have an exact sequence of graded Lie algebras

$$0 \longrightarrow \mathcal{N}_r^{(r)} \longrightarrow \mathcal{L}_g^{(r)} \xrightarrow{p_r^{(r)}} \mathcal{L}_g^{(r-1)} \longrightarrow 0. \tag{1.2.2}$$

1.3. We have also the "inserting the r-th string" homomorphism

$$\begin{aligned} (1.3.1)\quad & i = i^{(r-1)} \colon \mathcal{L}_g^{(r-1)} \longrightarrow \mathcal{L}_g^{(r)}, \\ & X_{ik}^{(r-1)}, Y_{ik}^{(r-1)}, Z_{jk}^{(r-1)} \longmapsto X_{ik}^{(r)}, Y_{ik}^{(r)}, Z_{jk}^{(r)} \quad (1 \leq i \leq g; 1 \leq j,k \leq r-1), \\ & Z_{0k}^{(r-1)} \longmapsto Z_{rk}^{(r)} + Z_{0k}^{(r)} \quad (1 \leq k \leq r-1), \end{aligned}$$

which is a splitting homomorphism of $p_r^{(r)}$. Any element in the image $i^{(r-1)}(\mathcal{L}_g^{(r-1)})$ commutes with $Z_{0r}^{(r)}$. Conversely we have

LEMMA 1.4: *The centralizer $C_{\mathcal{L}_g^{(r)}}(Z_{0r}^{(r)})$ of $Z_{0r}^{(r)}$ in $\mathcal{L}_g^{(r)}$* is

$$C_{\mathcal{L}_g^{(r)}}(Z_{0r}^{(r)}) = \mathcal{L}_{g,<r}^{(r)} \oplus \langle Z_{0r}^{(r)} \rangle.$$

Proof: Put $C = C_{\mathcal{L}_g^{(r)}}(Z_{0r}^{(r)})$ and $C' = \mathcal{L}_{g,<r}^{(r)} \oplus \langle Z_{0r}^{(r)} \rangle$. Then we have

$$(1.4.1)\qquad \begin{tikzcd} 0 \arrow[r] & \mathcal{N}_r^{(r)} \arrow[r] & \mathcal{L}_g^{(r)} \arrow[r, "p_r^{(r)}"] & \mathcal{L}_g^{(r-1)} \arrow[r] & 0 \\ 0 \arrow[r] & \langle Z_{0r}^{(r)} \rangle \arrow[r] \arrow[u, hook] & C \arrow[r] \arrow[u, hook] & p_r^{(r)}(C) \arrow[r] \arrow[u, hook] & 0 \\ 0 \arrow[r] & \langle Z_{0r}^{(r)} \rangle \arrow[r] \arrow[u, equal] & C' \arrow[r] \arrow[u, hook] & p_r^{(r)}(C') \arrow[r] \arrow[u, hook] & 0. \end{tikzcd}$$

Since $p_r^{(r)}(C') = \mathcal{L}_g^{(r-1)}$, the inclusions in the right column are equalities, hence $C_{\mathcal{L}_g^{(r)}}(Z_{0r}^{(r)}) = C'$. ∎

By this lemma, the homomorphism $i^{(r-1)}: \mathcal{L}_g^{(r-1)} \to \mathcal{L}_g^{(r)}$ is characterized as the unique (modulo $\langle Z_{0r}^{(r)} \rangle$ in degree two) splitting homomorphism of $p_r^{(r)}$ whose image commutes with $Z_{0r}^{(r)}$.

1.5. A linear endomorphism D of $\mathcal{L}_g^{(r)}$ is called a derivation if it satisfies the Leibniz rule $D([X,Y]) = [D(X),Y] + [X,D(Y)]$ for any $X, Y \in \mathcal{L}_g^{(r)}$. The set $\operatorname{Der}\mathcal{L}_g^{(r)}$ of all derivations of $\mathcal{L}_g^{(r)}$ forms a Lie algebra with respect to the bracket $[D, D'] = D \circ D' - D' \circ D$. Now we define the **braid-like derivation algebra** of $\mathcal{L}_g^{(r)}$:

$$(1.5.1)\qquad \operatorname{Der}^\flat \mathcal{L}_g^{(r)} = \left\{ D \in \operatorname{Der}\mathcal{L}_g^{(r)} \;\middle|\; \begin{array}{c} D(\mathcal{N}_k^{(r)}) \subset \mathcal{N}_k^{(r)} \ (1 \le k \le r) \\ D(Z_{jk}^{(r)}) = [U_{jk}^{(r)}, Z_{jk}^{(r)}] \\ (\exists U_{jk}^{(r)} \in \mathcal{L}_g^{(r)\circ}, 0 \le j \le r, 1 \le k \le r) \end{array} \right\}.$$

A braid-like derivation D is called **of degree** m if $D(\operatorname{gr}^i \mathcal{L}_g^{(r)}) \subset \operatorname{gr}^{i+m} \mathcal{L}_g^{(r)}$. We denote the submodule consisting of all the braid-like derivations of degree m by $\operatorname{gr}^m \operatorname{Der}^\flat \mathcal{L}_g^{(r)}$. Then, since $\mathcal{L}_g^{(r)}$ is generated by finitely many homogeneous elements, $\operatorname{Der}^\flat \mathcal{L}_g^{(r)}$ admits a decomposition into a direct sum of homogeneous components; $\operatorname{Der}^\flat \mathcal{L}_g^{(r)} = \bigoplus_{m=1}^\infty \operatorname{gr}^m \operatorname{Der}^\flat \mathcal{L}_g^{(r)}$. The set $\operatorname{Int}\mathcal{L}_g^{(r)}$ of all **inner derivations** $\operatorname{Int}(V) = [V, *]$ ($V \in \mathcal{L}_g^{(r)}$) forms a homogeneous Lie ideal of $\operatorname{Der}\mathcal{L}_g^{(r)}$. The quotient algebra

$$(1.5.2)\qquad \begin{aligned} \operatorname{Out}^\flat \mathcal{L}_g^{(r)} &= (\operatorname{Der}^\flat \mathcal{L}_g^{(r)} + \operatorname{Int}\mathcal{L}_g^{(r)}) / \operatorname{Int}\mathcal{L}_g^{(r)} \\ &= \operatorname{Der}^\flat \mathcal{L}_g^{(r)} / (\operatorname{Der}^\flat \mathcal{L}_g^{(r)} \cap \operatorname{Int}\mathcal{L}_g^{(r)}) \end{aligned}$$

is called the **outer braid-like derivation algebra**. We shall show later that $\operatorname{Der}^\flat \mathcal{L}_g^{(r)} \cap \operatorname{Int}\mathcal{L}_g^{(r)} = \operatorname{Int}\mathcal{L}_g^{(r)\circ}$.

Remark 1.6: This definition is slightly modified from that of [NTaU]. We require that $U_{jk}^{(r)}$ should be taken in $\mathcal{L}_g^{(r)\circ}$. Owing to Matsumoto's result [M], this is a reasonable restriction when we are interested in Galois representation. As a result of this definition, if $D \in \operatorname{Der} \mathcal{L}_g^{(1)}$ is braid-like then $D(Z_{01}^{(1)}) = 0$ because $\mathcal{L}_g^{(1)\circ} = \langle Z_{01}^{(1)} \rangle$.

1.7. The "forgeting the r-th string" homomorphism $p_r^{(r)}\colon \mathcal{L}_g^{(r)} \to \mathcal{L}_g^{(r-1)}$ naturally induces homomorphisms $\operatorname{Der}^\flat \mathcal{L}_g^{(r)} \to \operatorname{Der}^\flat \mathcal{L}_g^{(r-1)}$ and $\operatorname{Out}^\flat \mathcal{L}_g^{(r)} \to \operatorname{Out}^\flat \mathcal{L}_g^{(r-1)}$ since the kernel $\mathcal{N}_r^{(r)}$ of $p_r^{(r)}$ is preserved by $\operatorname{Der}^\flat \mathcal{L}_g^{(r)}$. We denote these homomorphisms by the same symbol $\psi_r^{(r)}$. We have the following remarkable property called "filtered injectivity" (Nakamura-Takao-Ueno [NTaU], see also a preceding work of Ihara-Kaneko [IK]).

THEOREM 1.8 (Nakamura-Takao-Ueno [NTaU] Theorem 4.3): *For any $g \geq 1$ and $r \geq 2$, the homomorphism*

$$\psi_r^{(r)}\colon \operatorname{Out}^\flat \mathcal{L}_g^{(r)} \longrightarrow \operatorname{Out}^\flat \mathcal{L}_g^{(r-1)}$$

is injective.

The proof of this theorem seems complicated. The author hopes that an easier proof will be found.

1.9. Put

$$W_k = W_k^{(r)} = \sum_{j=k+1}^{r} Z_{jk}^{(r)} + Z_{0k}^{(r)} \quad (1 \leq k \leq r) \tag{1.9.1}$$

(in particular, $W_r = W_r^{(r)} = Z_{0r}^{(r)}$). A braid-like derivation $D \in \operatorname{Der}^\flat \mathcal{L}_g^{(r)}$ is said to be W-**normalized** if $D(W_k^{(r)}) = 0$ for $1 \leq k \leq r$. Denote by $\operatorname{Der}^* \mathcal{L}_g^{(r)}$ the subalgebra consisting of all W-normalized derivations:

$$\operatorname{Der}^* \mathcal{L}_g^{(r)} = \{D \in \operatorname{Der}^\flat \mathcal{L}_g^{(r)} \mid D(W_k^{(r)}) = 0 \ (1 \leq k \leq r)\}. \tag{1.9.2}$$

The following proposition is an analogue of "y-normalization" ([I3] §2 Proposition 1).

PROPOSITION 1.10:

(1) *All $W_k^{(r)}$ $(1 \leq k \leq r)$ are mutually commutative.*

(2) $C_{\mathcal{L}_g^{(r)}}\left(\left\{W_k^{(r)} \mid l \leq k \leq r\right\}\right) = \mathcal{L}_{g,<l}^{(r)} \oplus \left\langle W_k^{(r)} \mid 1 \leq k \leq r\right\rangle$. *In particular, if $V \in \mathcal{L}_g^{(r)}$ commutes with all $W_k^{(r)}$'s, then $V \in \left\langle W_k^{(r)} \mid 1 \leq k \leq r\right\rangle$.*

(3) *If* $D \in \mathrm{Der}^\flat \mathcal{L}_g^{(r)}$ *satisfies* $D(W_k^{(r)}) \in \mathcal{N}_r^{(r)}$ *and* $D(W_r^{(r)}) = 0$, *then* $D(W_k^{(r)}) = 0$. *In particular, if* $D(W_k^{(r)}) \in \mathcal{N}_r^{(r)}$ *for all* $1 \le k \le r-1$ *and* $D(W_r^{(r)}) = 0$, *then* D *is* W*-normalized.*

(4) *For any* $D \in \mathrm{Der}^\flat \mathcal{L}_g^{(r)}$, *there exists* $V \in \mathcal{L}_g^{(r)\circ}$ *such that* $D - \mathrm{Int}(V) \in \mathrm{Der}^* \mathcal{L}_g^{(r)}$. *The element* $V \in \mathcal{L}_g^{(r)\circ}$ *is unique modulo* $\langle \mathrm{Int}(W_k^{(r)}) | \, 1 \le k \le r\rangle$.

(5) $\mathrm{Der}^\flat \mathcal{L}_g^{(r)} \cap \mathrm{Int}\,\mathcal{L}_g^{(r)} = \mathrm{Int}\,\mathcal{L}_g^{(r)\circ}$.

Proof: (1) For $1 \le k < l \le r$,

$$[W_k^{(r)}, W_l^{(r)}] = \left[-\sum_{i=1}^{g}[X_{ik}^{(r)}, Y_{ik}^{(r)}] - \sum_{j=1}^{k-1} Z_{jk}^{(r)}, \sum_{j=l+1}^{r} Z_{jl}^{(r)} + Z_{0l}^{(r)} \right] = 0.$$

(2) Induction on r. When $l = r$, it follows from Lemma 1.4. In particular, it holds when $r = l = 1$. When $l < r$, put $C_l^{(r)} = C_{\mathcal{L}_g^{(r)}}(\{W_k^{(r)} | \, l \le k \le r\})$ and $C_l^{(r)\prime} = \mathcal{L}_{g,<l}^{(r)} \oplus \langle W_k^{(r)} | \, 1 \le k \le r\rangle$. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N}_r^{(r)} \arrow[r] & \mathcal{L}_g^{(r)} \arrow[r, "p_r^{(r)}"] & \mathcal{L}_g^{(r-1)} \arrow[r] & 0 \\
0 \arrow[r] & \langle Z_{0r}^{(r)} \rangle \arrow[r] \arrow[u, hook] & C_l^{(r)} \arrow[r] \arrow[u, hook] & p_r^{(r)}(C_l^{(r)}) \arrow[r] \arrow[u, hook] & 0 \\
0 \arrow[r] & \langle Z_{0r}^{(r)} \rangle \arrow[r] \arrow[u, equal] & C_l^{(r)\prime} \arrow[r] \arrow[u, hook] & p_r^{(r)}(C_l^{(r)\prime}) \arrow[r] \arrow[u, hook] & 0,
\end{tikzcd}$$

where each row is exact. Since $p_r^{(r)}(W_k^{(r)}) = W_k^{(r-1)}$ $(1 \le k \le r-1), = 0$ $(k = r)$, it holds that $p_r^{(r)}(C_l^{(r)}) \subset C_l^{(r-1)}$ and that $p_r^{(r)}(C_l^{(r)\prime}) = C_l^{(r-1)\prime}$. By assumption of induction we have $C_l^{(r-1)} = C_l^{(r-1)\prime}$. Hence, by the diagram above, we obtain $p_r^{(r)}(C_l^{(r)}) = p_r^{(r)}(C_l^{(r)\prime})$ and $C_l^{(r)} = C_l^{(r-1)}$. In particular, for $l = 1$, we have $C_1^{(r)} = \langle W_k^{(r)} | \, 1 \le k \le r\rangle$.

(3) Since $W_k^{(r)}$ and $W_r^{(r)}$ commutes with each other, we have

$$0 = D([W_k^{(r)}, W_r^{(r)}]) = [D(W_k^{(r)}), W_r^{(r)}].$$

Hence $D(W_k^{(r)})$ belongs to $C_{\mathcal{N}_r^{(r)}}(W_r^{(r)}) = \langle W_r^{(r)} \rangle$. Since $D(W_k^{(r)})$ has its homogeneous components only in degree greater than two, it follows that $D(W_k^{(r)}) = 0$.

(4) Induction on r. Let $D \in \mathrm{Der}^\flat \mathcal{L}_g^{(r)}$ be a braid-like derivation. When $r = 1$, it is already normalized because $D(W_1^{(1)}) = D(Z_{01}^{(1)}) = 0$ (Remark 1.6). Let $r > 2$. Since $D(W_r^{(r)}) = D(Z_{0r}^{(r)}) = [U_{0r}^{(r)}, Z_{0r}^{(r)}]$ with $U_{0r}^{(r)} \in \mathcal{L}_g^{(r)\circ}$, we can assume

that $D(W_r^{(r)}) = 0$ without loss of generality by replacing D with $D - \mathrm{Int}(U_{0r}^{(r)})$. Let $D' = \psi_r^{(r)}(D)$ be the induced derivation on $\mathcal{L}_g^{(r-1)}$. By the assumption of induction, there exist $V' \in \mathcal{L}_g^{(r-1)\circ}$ such that $D' - \mathrm{Int}(V') \in \mathrm{Der}^* \mathcal{L}_g^{(r-1)}$. Put $V = i^{(r-1)}(V') \in C_{\mathcal{L}_g^{(r)\circ}}(W_r^{(r)})$ and $D^* = D - \mathrm{Int}(V)$. Then $D^*(W_r^{(r)}) = D(W_r^{(r)}) - [V, W_r^{(r)}] = 0$, and $D^*(W_k^{(r)}) \in \mathcal{N}_r^{(r)}$ for $1 \leq k \leq r-1$ because $p_r^{(r)}(D^*(W_k^{(r)})) = (D' - \mathrm{Int}(V'))(W_k^{(r-1)}) = 0$. By (3) we have $D^* \in \mathrm{Der}^* \mathcal{L}_g^{(r)}$. Uniqueness follows from (2).

(5) Suppose that $U \in \mathcal{L}_g^{(r)}$ satisfies $\mathrm{Int}(U) \in \mathrm{Der}^\flat \mathcal{L}_g^{(r)}$. Then, by (4), there exist $V \in \mathcal{L}_g^{(r)\circ}$ such that $\mathrm{Int}(U) - \mathrm{Int}(V) \in \mathrm{Der}^* \mathcal{L}_g^{(r)}$. Then $U - V$ commutes with $W_k^{(r)}$ $(1 \leq k \leq r)$. By (2), $U - V$ belongs to $\langle W_k^{(r)} | \, 1 \leq k \leq r \rangle$. Hence $U \in \mathcal{L}_g^{(r)\circ}$. ∎

By this proposition, we have

$$\begin{aligned} \mathrm{Out}^\flat \mathcal{L}_g^{(r)} &= \mathrm{Der}^\flat \mathcal{L}_g^{(r)} / (\mathrm{Der}^\flat \mathcal{L}_g^{(r)} \cap \mathrm{Int}\, \mathcal{L}_g^{(r)}) = \mathrm{Der}^\flat \mathcal{L}_g^{(r)} / \mathrm{Int}\, \mathcal{L}_g^{(r)\circ} \\ &\simeq \mathrm{Der}^* \mathcal{L}_g^{(r)} / \left\langle \mathrm{Int}(W_k^{(r)}) | \, 1 \leq k \leq r \right\rangle. \end{aligned} \tag{1.10.1}$$

1.11. Since $p_r^{(r)}(W_k^{(r)}) = W_k^{(r-1)}$ $(1 \leq k \leq r-1), = 0$ $(k = r)$, the natural homomorphism $\mathrm{Der}^\flat \mathcal{L}_g^{(r)} \to \mathrm{Der}^\flat \mathcal{L}_g^{(r-1)}$ defines

$$\mathrm{Der}^* \mathcal{L}_g^{(r)} \to \mathrm{Der}^* \mathcal{L}_g^{(r-1)}. \tag{1.11.1}$$

The kernel of this morphism is $\langle \mathrm{Int}(W_r^{(r)}) \rangle$ since $\mathrm{Out}^\flat \mathcal{L}_g^{(r)} \to \mathrm{Out}^\flat \mathcal{L}_g^{(r-1)}$ is injective. In particular, it is injective on the components of degree greater than two.

1.12. We denote by $\mathcal{D}_g^{(r)}$ the subalgebra of $\mathrm{Der}^* \mathcal{L}_g^{(r)}$ consisting of all W-normalized derivations which are $\mathfrak{S}_r$-invariant modulo inner derivations. By (1.11.1) we have a sequence of homomorphisms

$$\cdots \longrightarrow \mathcal{D}_g^{(r)} \longrightarrow \cdots \longrightarrow \mathcal{D}_g^{(3)} \longrightarrow \mathcal{D}_g^{(2)} \longrightarrow \mathcal{D}_g^{(1)} = \mathrm{Der}^* \mathcal{L}_g^{(1)}, \tag{1.12.1}$$

which are injective up to one-dimensinal kernels in degree two. A derivation $D \in \mathcal{D}_g^{(r)}$ is said to be **stable** if it lies in the image of $\mathcal{D}_g^{(r')}$ for any $r' \geq r$. In other words, a stable derivation is a derivation of $\mathcal{L}_g^{(r)}$ which can be lifted to an element of $\mathcal{D}_g^{(r')}$ for any $r' \geq r$. Our interest is the determination of the subalgebra consisting of all the stable derivations — the **stable derivation algebra**.

Now we shall point out a remarkable trick on $\mathfrak{S}_r$-invariance of W-normalized derivations, whose proof depends essentially on Theorem 1.8 (filtered injectivity = [NTaU] Theorem 4.3).

PROPOSITION 1.13: *For $r \geq 3$, the action of $\mathfrak{S}_r$ on $\mathrm{Out}^\flat \mathcal{L}_g^{(r)}$ is trivial, that is, every element of* $\mathrm{Der}^\flat \mathcal{L}_g^{(r)}$ *is automatically $\mathfrak{S}_r$-invariant modulo inner derivations.*

Proof: It suffices to show that every transposition acts trivially on $\mathrm{Out}^\flat \mathcal{L}_g^{(r)}$. Since any two transpositions are mutually conjugate, it suffices only to see the action of transposition $\sigma = (r-1\ r)$. Since the two-step composite $p_r^{(r)} \circ p_{r-1}^{(r-1)}\colon \mathcal{L}_g^{(r)} \to \mathcal{L}_g^{(r-2)}$ of the string-forgetting maps commutes with σ, also the induced homomorphism $\psi_r^{(r)} \circ \psi_{r-1}^{(r-1)}\colon \mathrm{Out}^\flat \mathcal{L}_g^{(r)} \to \mathrm{Out}^\flat \mathcal{L}_g^{(r-2)}$ commutes with the action of σ. Then the triviality of the action of σ on $\mathrm{Out}^\flat \mathcal{L}_g^{(r)}$ follows from the injectivity of $\psi_r^{(r)} \circ \psi_{r-1}^{(r-1)}$ (Theorem 1.8) ∎

1.14. To consider the stability in our case of higher genera, we want to combine our situation with the study of the case of genus zero. The center $C(\mathcal{L}_g^{(r)\circ})$ of $\mathcal{L}_g^{(r)\circ}$ is generated by $\sum_{0\leq j<k\leq r} Z_{jk}^{(r)} = \sum_{1\leq k\leq r} W_k^{(r)}$. We have a natural isomorphism

$$\mathcal{L}_g^{(r)\circ}/C(\mathcal{L}_g^{(r)\circ}) \xrightarrow{\sim} \mathfrak{P}^{(r+2)} \tag{1.14.1}$$

mapping $Z_{jk}^{(r)}$ to $x_{j+1,k+1}^{(r+2)}$. It sends the component of degree $2m$ to that of degree m and maps $W_k^{(r)}$'s to $y_{k+1}^{(r+2)}$ up to sign. See the Appendix for notation in the case of genus zero.

LEMMA 1.15:
(1) *Let $D \in \mathcal{D}^{(2)}$. Then the restriction D_0 on $\mathcal{L}_g^{(2)\circ}$ can be identified with an element of* $\mathrm{Out}^\flat \mathfrak{P}^{(4)}$ *fixed by the action of* $(2\ 3) \in \mathfrak{S}_4$ *via (1.14.1).*
(2) *Let $r \geq 3$ and $D \in \mathcal{D}^{(r)}$. Then the restriction D_0 on $\mathcal{L}_g^{(r)\circ}$ can be identified with an element of $\mathcal{D}_0^{(r+2)}$ via (1.14.1).*

Proof: The conditions on braid-likeness and normalization are satisfied by definition. The $\mathfrak{S}_{r+2}$-invariance (up to an inner derivation) is the only matter to check. If $r=2$, D is invariant under a transposition $(1\ 2)$. Hence D_0 is invariant under a transposition $(2\ 3)$. If $r=3$, D_0 is automatically $\mathfrak{A}_5$-invariant. Since D is invariant under $\mathfrak{S}_3$, in particular under a transposition, D_0 is $\mathfrak{S}_5$-invariant. If $r \geq 4$, D_0 is automatically $\mathfrak{S}_{r+2}$-invariant (see Proposition A.10). ∎

The following proposition is the key proposition to prove our main theorem. The proof is given in the next section.

PROPOSITION 1.16: *Let $D = D^{(2)} \in \mathcal{D}_g^{(2)}$ and identify the restriction $D_0 = D_0^{(2)}$ on $\mathcal{L}_g^{(2)\circ}$ with an element of $\mathrm{Out}^\flat \mathfrak{P}^{(4)}$. For $r \geq 3$, if $D_0^{(2)}$ can be lifted to $D_0^{(r)} \in \mathcal{D}_0^{(r+2)}$, then $D^{(2)}$ is stable, that is, $D^{(2)}$ can be lifted to an element $D^{(r)} \in \mathcal{D}_g^{(r)}$.*

Our main theorem follows from the proposition above combined with Ihara's result [I3] for the stability in the case of genus zero.

THEOREM 1.17: *If $D = D^{(1)} \in \mathcal{D}_g^{(1)}$ can be lifted to $D^{(3)} \in \mathcal{D}_g^{(3)}$, then $D^{(1)}$ is stable. Therefore we have*

$$\cdots \xrightarrow{\sim} \mathcal{D}_g^{(r)} \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{D}_g^{(4)} \xrightarrow{\sim} \mathcal{D}_g^{(3)} \hookrightarrow \mathcal{D}_g^{(2)} \hookrightarrow \mathcal{D}_g^{(1)}.$$

In other words, $\mathcal{D}_g^{(3)}$ is the **stable derivation algebra** *for genus g.*

Proof: By Lemma 1.15 the restriction of $D^{(3)}$ on $\mathcal{L}_g^{(3)\circ}$ defines an element of $\mathcal{D}_0^{(5)}$. Since any element of $\mathcal{D}_0^{(5)}$ can be lifted in $\mathcal{D}_0^{(r+2)}$ for all $r \geq 3$ ([I3]), the conclusion follows from the proposition above. ∎

COROLLARY 1.18: *If $D = D^{(2)} \in \mathcal{D}_g^{(2)}$ is 0-map on $\mathcal{L}_g^{(2)\circ}$ (i.e., $U_{jk}^{(2)} = 0$ for all j, k), then D is stable.*

2. Proof of the key proposition

In this section we shall give a proof of Proposition 1.16.

For a derivation $D = D^{(1)} \in \mathcal{D}_g^{(1)}$, put $S_{i1}^{(1)} = D^{(1)}(X_{i1}^{(1)}), T_{i1}^{(1)} = D^{(1)}(Y_{i1}^{(1)})$ for $1 \leq i \leq g$. Since $\mathcal{L}_g^{(1)}$ is a free Lie algebra on $X_{j1}^{(1)}, Y_{j1}^{(1)}$ $(1 \leq j \leq g)$, $S_{i1}^{(1)}$ and $T_{i1}^{(1)}$ are Lie polynomials of $2g$ variables. In this sense, we employ the notation for Lie polynomials of $2g$ variables $X_{j1}^{(1)}, Y_{j1}^{(1)}$ $(1 \leq j \leq g)$ as $S_{i1}^{(1)} =: \mathcal{S}_i(\mathbf{X}_1^{(1)}, \mathbf{Y}_1^{(1)}), T_{i1}^{(1)} =: \mathcal{T}_i(\mathbf{X}_1^{(1)}, \mathbf{Y}_1^{(1)})$, where $\mathbf{X}_1^{(1)} = (X_{11}^{(1)}, \ldots, X_{g1}^{(1)})$ and $\mathbf{Y}_1^{(1)} = (Y_{11}^{(1)}, \ldots, Y_{g1}^{(1)})$ are multi-variables. For multi-variables $\mathbf{X}_k^{(r)} = (X_{1k}^{(r)}, \ldots, X_{gk}^{(r)})$ and $\mathbf{Y}_k^{(r)} = (Y_{1k}^{(r)}, \ldots, Y_{gk}^{(r)})$, we also denote by $\mathcal{S}_i(\mathbf{X}_k^{(r)}, \mathbf{Y}_k^{(r)})$ (resp. $\mathcal{T}_i(\mathbf{X}_k^{(r)}, \mathbf{Y}_k^{(r)})$) the element of $\mathcal{L}_g^{(r)}$ obtained by replacing $X_{j1}^{(1)}$'s and $Y_{j1}^{(1)}$'s $(1 \leq j \leq g)$ in $S_{i1}^{(1)}$ (resp. $T_{i1}^{(1)}$) with $X_{jk}^{(r)}$'s and $Y_{jk}^{(r)}$'s. Since D is normalized to be $D(Z_1^{(1)}) = 0$, we have

$$\sum_{i=1}^{g}([S_{i1}^{(1)}, Y_{i1}^{(1)}] + [X_{i1}^{(1)}, T_{i1}^{(1)}]) = 0. \tag{2.0.1}$$

LEMMA 2.1: *Assume that $D^{(1)}$ can be lifted to $D^{(2)} \in \mathcal{D}_g^{(2)}$ and put $D^{(2)}(X_{ik}^{(2)}) =: S_{ik}^{(2)}, D^{(2)}(Y_{ik}^{(2)}) =: T_{ik}^{(2)}$ $(k = 1,2)$, $D^{(2)}(Z_{jk}^{(2)}) =: [U_{jk}^{(2)}, Z_{jk}^{(2)}]$ $(U_{jk}^{(2)} \in \mathcal{L}_g^{(2)\circ}$, $(j,k) = (0,1),(0,2),(1,2))$. Then the following equations hold:*

$$
\begin{aligned}
S_{i1}^{(2)} &= \mathcal{S}_i(\mathbf{X}_1^{(2)}, \mathbf{Y}_1^{(2)}),\\
T_{i1}^{(2)} &= \mathcal{T}_i(\mathbf{X}_1^{(2)}, \mathbf{Y}_1^{(2)}),\\
S_{i2}^{(2)} &= \mathcal{S}_i(\mathbf{X}_2^{(2)}, \mathbf{Y}_2^{(2)}) + [U_{01}^{(2)}, X_{i2}^{(2)}],\\
T_{i2}^{(2)} &= \mathcal{T}_i(\mathbf{X}_1^{(2)}, \mathbf{Y}_1^{(2)}) + [U_{01}^{(2)}, Y_{i2}^{(2)}],\\
U_{02}^{(2)} &= 0,\\
U_{01}^{(2)} &= U_{12}^{(2)} - \sigma(U_{12}^{(2)}) \quad (\sigma = (1\ 2) \in \mathfrak{S}_2).
\end{aligned}
$$

Proof: We have $D^{(2)}(Z_{02}^{(2)}) = 0$ by the condition of W-normalization. Operating $D^{(2)}$ on $[X_{i1}^{(2)}, Z_{02}^{(2)}] = 0$, we have $[S_{i1}^{(2)}, Z_{02}^{(2)}] = 0$. From this and $p_2^{(2)}(S_{i1}^{(2)}) = S_{i1}^{(1)}$, it follows that $S_{i1}^{(2)} = i^{(1)}(S_{i1}^{(1)}) = \mathcal{S}_i(\mathbf{X}_1^{(2)}, \mathbf{Y}_1^{(2)})$. Similarly we get $T_{i1}^{(2)} = \mathcal{T}_i(\mathbf{X}_1^{(2)}, \mathbf{Y}_1^{(2)})$. Then consider $\sigma = (1\ 2)$-invariance modulo inner derivations. Since $\sigma \circ D^{(2)} \circ \sigma^{-1}(Z_{02}^{(2)}) = [\sigma(U_{01}^{(2)}), Z_{02}^{(2)}]$, $\sigma \circ D^{(2)} \circ \sigma^{-1}$ must coincide with $D^{(2)} + \mathrm{Int}(\sigma(U_{01}^{(2)}))$. The rest follows from this. Notice that $U_{01}^{(2)} = U_{12}^{(2)} - \sigma(U_{12}^{(2)})$ implies $\sigma(U_{01}^{(2)}) = -U_{01}^{(2)}$. ∎

Let $D_0^{(2)}$ be the restriction of $D^{(2)}$ on $\mathcal{L}_g^{(2)\circ}$. We assume that $D_0^{(2)}$ can be extended to $D_0^{(r)}$ on $\mathcal{L}_g^{(r)\circ}$. We put $D_0^{(r)}(Z_{jk}^{(r)}) = [U_{jk}^{(r)}, Z_{jk}^{(r)}]$ for $j \in \{1, \ldots, r\} \cup \{0\}$ and $k = 1, \ldots, r$. By $\mathfrak{S}_r$-symmetry of $D_0^{(r)}$, there is a 1-cocycle $a^{(r)}\colon \mathfrak{S}_r \to \mathcal{L}_g^{(r)\circ}$ satisfying $\sigma \circ D_0^{(r)} \circ \sigma^{-1} = D_0^{(r)} + \mathrm{Int}(a^{(r)}(\sigma))$ for $\sigma \in \mathfrak{S}_r$. We desire to construct $D^{(r)} \in \mathcal{D}_g^{(r)}$ which lifts $D^{(2)}$ and whose restriction on $\mathcal{L}_g^{(r)\circ}$ coincides with $D_0^{(r)}$.

CLAIM 2.2: *The desired derivation $D^{(r)} \in \mathcal{D}_g^{(r)}$ is given by the following assignment:*

$$X_{ik}^{(r)} \longmapsto S_{ik}^{(r)} := \mathcal{S}_i(\mathbf{X}_k^{(r)}, \mathbf{Y}_k^{(r)}) - [a^{(r)}(\tau_k), X_{ik}^{(r)}], \tag{2.2.1}$$

$$Y_{ik}^{(r)} \longmapsto T_{ik}^{(r)} := \mathcal{T}_i(\mathbf{X}_k^{(r)}, \mathbf{Y}_k^{(r)}) - [a^{(r)}(\tau_k), Y_{ik}^{(r)}], \tag{2.2.2}$$

$$Z_{jk}^{(r)} \longmapsto D_0^{(r)}(Z_{jk}^{(r)}) = [U_{jk}^{(r)}, Z_{jk}^{(r)}]. \tag{2.2.3}$$

Here τ_k is any element of $\mathfrak{S}_r$ with $\tau_k(1) = k$ and $S_{ik}^{(r)}$ and $T_{ik}^{(r)}$ do not depend on the choice of τ_k.

Proof: We begin with showing the following lemma which assures that the elements $S_{ik}^{(r)}$ and $T_{ik}^{(r)}$ in the definition of $D^{(r)}$ do not depend on the choice of τ_k.

LEMMA 2.3: *If $\sigma \in \mathfrak{S}_r$ satisfies $\sigma(1) = 1$, then $a^{(r)}(\sigma)$ commutes with $W_1^{(r)}$, hence belongs to $\mathcal{L}_{g,\geq 2}^{(r)\circ} := \langle Z_{jk}^{(r)} \mid j,k \geq 2\rangle$. Therefore it commutes also with $X_{i1}^{(r)}$ and $Y_{i1}^{(r)}$ $(i = 1, \ldots, g)$.*

Proof: If $\sigma(1) = 1$, σ fixes $W_1^{(r)}$. The element $a^{(r)}(\sigma)$ satisfies, by definition, $D_0^{(r)} = \sigma \circ D_0^{(r)} \circ \sigma^{-1} + \mathrm{Int}(a^{(r)}(\sigma))$. Therefore

$$[a^{(r)}(\sigma), W_1^{(r)}] = D_0^{(r)}(W_1^{(r)}) - \sigma \circ D_0^{(r)} \circ \sigma^{-1}(W_1^{(r)}) = 0. \quad \blacksquare$$

We shall prove this claim by induction on r. The case $r = 2$ is trivial. The homomorphism $p_r^{(r)}|_{\mathcal{L}_g^{(r)\circ}}\colon \mathcal{L}_g^{(r)\circ} \to \mathcal{L}_g^{(r-1)\circ}$ induces a derivation $D_0^{(r-1)}\colon \mathcal{L}_g^{(r-1)\circ} \to \mathcal{L}_g^{(r-1)\circ}$ from $D_0^{(r)}$. By the assumption of induction, there is a derivation $D^{(r-1)}\colon \mathcal{L}_g^{(r-1)\circ} \to \mathcal{L}_g^{(r-1)\circ}$ of the above form which extends $D_0^{(r-1)}$.

LEMMA 2.4:

(i) *When we identify $\mathfrak{S}_{r-1}$ with the stabilizer of the letter r in $\mathfrak{S}_r$, we have $p_r^{(r)}(a^{(r)}(\sigma)) = a^{(r-1)}(\sigma)$ for $\sigma \in \mathfrak{S}_{r-1}$.*

(ii) *When $\{j,k\} \cap \{r, 0\} = \emptyset$, we have $S_{ik}^{(r)} = \iota(S_{ik}^{(r-1)}), T_{ik}^{(r)} = \iota(T_{ik}^{(r-1)})$ and $U_{ik}^{(r)} = \iota(U_{ik}^{(r-1)})$.*

Proof: (i) We have $D_0^{(r)} = \sigma \circ D_0^{(r)} \circ \sigma^{-1} - \mathrm{Int}(a^{(r)}(\sigma))$. Since $\sigma \in \mathfrak{S}_{r-1}$ commutes with the projection $p_r^{(r)}\colon \mathcal{L}_g^{(r)} \to \mathcal{L}_g^{(r-1)}$, we obtain an equation

$$D_0^{(r-1)} = \sigma \circ D_0^{(r-1)} \circ \sigma^{-1} - \mathrm{Int}(p_r^{(r)}(a^{(r)}(\sigma)))$$

between induced derivations on $\mathcal{L}_g^{(r-1)}$. This equation characterizes $a^{(r-1)}(\sigma)$.

(ii) By definition, we have $S_{ik}^{(r)} = \mathcal{S}_i(\mathbf{X}_k^{(r)}, \mathbf{Y}_k^{(r)}) - [a^{(r)}(\tau_k), X_{ik}^{(r)}]$ $(\tau_k = (1\ k))$. Since $k \neq r$, τ_k belongs to $\mathfrak{S}_{r-1}$. Hence, by (i), it follows that $p_r^{(r)}(S_{ik}^{(r)}) = \mathcal{S}_i(\mathbf{X}_k^{(r-1)}, \mathbf{Y}_k^{(r-1)}) - [a^{(r-1)}(\tau_k), X_{ik}^{(r-1)}] = S_{ik}^{(r-1)}$. On the other hand, $S_{ik}^{(r)}$ commutes with $Z_{0r}^{(r)}$ because $\mathcal{S}_i(\mathbf{X}_k^{(r)}, \mathbf{Y}_k^{(r)})$ commutes with $Z_{0r}^{(r)}$ and

$$[a^{(r)}(\tau_k), Z_{0r}^{(r)}] = D_0^{(r)}(Z_{0r}^{(r)}) - \sigma \circ D_0^{(r)} \circ \sigma^{-1}(Z_{0r}^{(r)}) = 0.$$

These two properties characterize $\iota(S_{ik}^{(r-1)})$ when it is of degree greater than two. The proof for $T_{ik}^{(r)}$ is quite similar. $\blacksquare$

LEMMA 2.5: *For any $\sigma \in \mathfrak{S}_r$, we have*

$$\sigma(S_{ik}^{(r)}) = S_{i\sigma(k)}^{(r)} + [a^{(r)}(\sigma), X_{i\sigma(k)}^{(r)}], \tag{2.5.1}$$

$$\sigma(T_{ik}^{(r)}) = T_{i\sigma(k)}^{(r)} + [a^{(r)}(\sigma), Y_{i\sigma(k)}^{(r)}], \tag{2.5.2}$$

$$\sigma(U_{jk}^{(r)}) = U_{\sigma(j)\sigma(k)}^{(r)} + a^{(r)}(\sigma), \tag{2.5.3}$$

$$\sigma(U_{0k}^{(r)}) = U_{0\sigma(k)}^{(r)} + a^{(r)}(\sigma). \tag{2.5.4}$$

Proof: Since $D_0^{(r)} = \sigma \circ D_0^{(r)} \circ \sigma^{-1} + \mathrm{Int}(a^{(r)}(\sigma))$, the assertions on U-coordinates follow. For S-coordinates we have

$$\begin{aligned}
\sigma(S_{ik}^{(r)}) &= \sigma(\mathcal{S}_i(\mathbf{X}_k^{(r)}, \mathbf{Y}_k^{(r)}) - [a^{(r)}(\tau_k), X_{ik}^{(r)}]) \\
&= \mathcal{S}_i(\mathbf{X}_{\sigma(k)}^{(r)}, \mathbf{Y}_{\sigma(k)}^{(r)}) - [\sigma a^{(r)}(\tau_k), X_{i\sigma(k)}^{(r)}] \\
&= S_{i\sigma(k)}^{(r)} + [a^{(r)}(\tau_{\sigma(k)}), X_{ik}^{(r)}] - [a^{(r)}(\sigma\tau_k) - a^{(r)}(\sigma), X_{ik}^{(r)}].
\end{aligned}$$

Since $\tau_{\sigma(k)}(1) = \sigma\tau_k(1) = \sigma(k)$, we have $[a^{(r)}(\tau_{\sigma(k)}), X_{ik}^{(r)}] = [a^{(r)}(\sigma\tau_k),\ X_{ik}^{(r)}]$, hence $\sigma(S_{ik}^{(r)}) = S_{i\sigma(k)}^{(r)} + [a^{(r)}(\sigma), X_{i\sigma(k)}^{(r)}]$. The proof for T-coordinates is quite similar.

Our proof of Claim 2.2 is done by checking that $D^{(r)}$ preserves all the defining relations of $\mathcal{L}_g^{(r)}$. We have nothing to do with (1.1.1) and (1.1.3) because of the well-definedness of $D_0^{(r)}$.

To show that $D^{(r)}$ preserves the relation (1.1.2)

$$\sum_{i=1}^{g}[X_{ik}^{(r)}, Y_{ik}^{(r)}] + \sum_{j=0}^{r} Z_{jk}^{(r)} = 0 \quad (1 \le k \le r), \tag{1.1.2}$$

we need to show

$$\sum_{i=1}^{g}([X_{ik}^{(r)}, T_{ik}^{(r)}] + [S_{ik}^{(r)}, Y_{ik}^{(r)}]) + \sum_{j=0}^{r}[U_{jk}^{(r)}, Z_{jk}^{(r)}] = 0. \tag{2.5.5}$$

When $k = 1$, it holds since

$$\begin{aligned}
&\sum_{i=1}^{g}([X_{i1}^{(r)}, T_{i1}^{(r)}] + [S_{i1}^{(r)}, Y_{i1}^{(r)}]) \\
&= \sum_{i=1}^{g}([X_{i1}^{(r)}, \mathcal{T}_i(\mathbf{X}_1^{(r)}, \mathbf{Y}_1^{(r)})] + [\mathcal{S}_i(\mathbf{X}_1^{(r)}, \mathbf{Y}_1^{(r)}), Y_{i1}^{(r)}]) \\
&= \iota^{(r-1)} \circ \cdots \circ \iota^{(1)}\left(\sum_{i=1}^{g}([X_{i1}^{(1)}, \mathcal{T}_i(\mathbf{X}_1^{(1)}, \mathbf{Y}_1^{(1)})] + [\mathcal{S}_i(\mathbf{X}_1^{(1)}, \mathbf{Y}_1^{(1)}), Y_{i1}^{(1)}])\right) \\
&= 0
\end{aligned}$$

and

$$\sum_{j=0}^{r}[U_{j1}^{(r)}, Z_{j1}^{(r)}] = D_0^{(r)}(\sum_{j=0}^{r} Z_{j1}^{(r)}) = D_0^{(r)}(W_1^{(r)}) = 0.$$

For general k, taking $\sigma \in \mathfrak{S}_r$ with $\sigma(k) = 1$,

$$\begin{aligned}
&\sigma\Big(\sum_{i=1}^{g}([X_{ik}^{(r)}, T_{ik}^{(r)}] + [S_{ik}^{(r)}, Y_{ik}^{(r)}]) + \sum_{j=0}^{r}[U_{jk}^{(r)}, Z_{jk}^{(r)}]\Big)\\
&=\sum_{i=1}^{g}([X_{i1}^{(r)}, T_{i1}^{(r)} + [a^{(r)}(\sigma), Y_{i1}^{(r)}]] + [S_{i1}^{(r)} + [a^{(r)}(\sigma), X_{i1}^{(r)}], Y_{i1}^{(r)}])\\
&\quad+\sum_{j=0}^{r}[U_{j1}^{(r)} + a^{(r)}(\sigma), Z_{j1}^{(r)}]\\
&=\sum_{i=1}^{g}([X_{i1}^{(r)}, T_{i1}^{(r)}] + [S_{i1}^{(r)}, Y_{i1}^{(r)}]) + \sum_{j=0}^{r}[U_{j1}^{(r)}, Z_{j1}^{(r)}]\\
&\quad+[a^{(r)}(\sigma), \sum_{i=1}^{g}[X_{ik}^{(r)}, Y_{ik}^{(r)}] + \sum_{j=0}^{r} Z_{jk}^{(r)}]\\
&= 0.
\end{aligned}$$

For the relation (1.1.4)

$$[X_{ik}^{(r)}, Z_{jl}^{(r)}] = [Y_{ik}^{(r)}, Z_{jl}^{(r)}] = 0 \quad (k \neq j, l), \tag{1.1.4}$$

we need to show

$$[S_{ik}^{(r)}, Z_{jl}^{(r)}] + [X_{ik}^{(r)}, [U_{jl}^{(r)}, Z_{jl}^{(r)}]] = 0, \tag{2.5.6}$$

$$[T_{ik}^{(r)}, Z_{jl}^{(r)}] + [Y_{ik}^{(r)}, [U_{jl}^{(r)}, Z_{jl}^{(r)}]] = 0. \tag{2.5.7}$$

Here we shall show the former one. The latter is quite similar. We treat the cases $j = 0$ and $j = 1, \ldots, r$ separately. If $(j, k, l) = (0, 1, r)$, it holds since $S_{i1}^{(r)} = \mathcal{S}_i(\mathbf{X}_1^{(r)}, \mathbf{Y}_1^{(r)})$ and $Z_{0r}^{(r)}$ commute with each other and since $[U_{0r}^{(r)}, Z_{0r}^{(r)}] = D_0^{(r)}(Z_{0r}^{(r)}) = D_0^{(r)}(W_r^{(r)}) = 0$. When $j = 0$ and k, l are general, taking $\sigma \in \mathfrak{S}_r$ with $\sigma(k) = 1, \sigma(l) = r$,

$$\begin{aligned}
&\sigma([S_{ik}^{(r)}, Z_{jl}^{(r)}] + [X_{ik}^{(r)}, [U_{jl}^{(r)}, Z_{jl}^{(r)}]])\\
&= [S_{i1}^{(r)} + [a^{(r)}(\sigma), X_{i1}^{(r)}], Z_{0r}^{(r)}] + [X_{i1}^{(r)}, [U_{0r}^{(r)} + a^{(r)}(\sigma), Z_{0r}^{(r)}]]\\
&= [S_{i1}^{(r)}, Z_{0r}^{(r)}] + [X_{i1}^{(r)}, [U_{0r}^{(r)}, Z_{0r}^{(r)}]] + [a^{(r)}(\sigma), [X_{i1}^{(r)}, Z_{0r}^{(r)}]]\\
&= 0.
\end{aligned}$$

If $(j, k, l) = (r-1, 1, r)$ (notice that $r - 1 \neq 1$ because $r \geq 3$), first we observe that $0 = D_0^{(r)}(W_{r-1}^{(r)}) = D_0^{(r)}(Z_{r-1,r}^{(r)} + Z_{0,r-1}^{(r)}) = [U_{r-1,r}^{(r)}, Z_{r-1,r}^{(r)}] + [U_{0,r-1}^{(r)}, Z_{0,r-1}^{(r)}]$. Using this and the equation for $j = 0$, we have

$$\begin{aligned}
[X_{i1}^{(r)}, [U_{r-1,r}^{(r)}, Z_{r-1,r}^{(r)}]] &= -[X_{i1}^{(r)}, [U_{0,r-1}^{(r)}, Z_{0,r-1}^{(r)}]]\\
&= [S_{i1}^{(r)}, Z_{0,r-1}^{(r)}] = [\mathcal{S}_i(\mathbf{X}_1^{(r)}, \mathbf{Y}_1^{(r)}), Z_{0,r-1}^{(r)}] = 0.
\end{aligned}$$

The equation follows from this and that $[S_{i1}^{(r)}, Z_{r-1,r}^{(r)}] = [\mathcal{S}_i(\mathbf{X}_1^{(r)}, \mathbf{Y}_1^{(r)}), Z_{r-1,r}^{(r)}] = 0$. When $j \neq 0$ and k, l are general, taking $\sigma \in \mathfrak{S}_r$ with $\sigma(j) = r-1, \sigma(k) = 1, \sigma(l) = r$,

$$\begin{aligned}
&\sigma([S_{ik}^{(r)}, Z_{jl}^{(r)}] + [X_{ik}^{(r)}, [U_{jl}^{(r)}, Z_{jl}^{(r)}]]) \\
&= [S_{i1}^{(r)} + [a^{(r)}(\sigma), X_{i1}^{(r)}], Z_{r-i,r}^{(r)}] + [X_{i1}^{(r)}, [U_{r-1,r}^{(r)} + a^{(r)}(\sigma), Z_{r-1,r}^{(r)}]] \\
&= [S_{i1}^{(r)}, Z_{r-1,r}^{(r)}] + [X_{i1}^{(r)}, [U_{r-1,r}^{(r)}, Z_{r-1,r}^{(r)}]] + [a^{(r)}(\sigma), [X_{i1}^{(r)}, Z_{r-1,r}^{(r)}]] \\
&= 0.
\end{aligned}$$

For the relation (1.1.5)

$$[X_{ik}, X_{jl}] = [Y_{ik}, Y_{jl}] = 0 \quad (k \neq l), \tag{1.1.5}$$

we need to show that

$$[S_{ik}^{(r)}, X_{jl}^{(r)}] + [X_{ik}^{(r)}, S_{jl}^{(r)}] = 0, \tag{2.5.8}$$

$$[T_{ik}^{(r)}, Y_{jl}^{(r)}] + [Y_{ik}^{(r)}, T_{jl}^{(r)}] = 0. \tag{2.5.9}$$

Here again we shall show only the former. If $(k, l) = (1, 2)$, we have

$$[S_{i1}^{(r)}, X_{j2}^{(r)}] + [X_{i1}^{(r)}, S_{j2}^{(r)}] = \iota^{(r-1)}([S_{i1}^{(r-1)}, X_{j2}^{(r-1)}] + [X_{i1}^{(r-1)}, S_{j2}^{(r-1)}]) = 0$$

by the assumption of induction (notice that $r > 2$). For general k and l, taking $\sigma \in \mathfrak{S}_r$ with $\sigma(k) = 1, \sigma(l) = 2$,

$$\begin{aligned}
&\sigma([S_{ik}^{(r)}, X_{jl}^{(r)}] + [X_{ik}^{(r)}, S_{jl}^{(r)}]) \\
&= [S_{i1}^{(r)} + [a^{(r)}(\sigma), X_{i1}^{(r)}], X_{j2}^{(r)}] + [X_{i1}^{(r)}, S_{j2}^{(r)} + [a^{(r)}(\sigma), X_{i2}^{(r)}]] \\
&= [S_{i1}^{(r)}, X_{j2}^{(r)}] + [X_{i1}^{(r)}, S_{j2}^{(r)}] + [a^{(r)}(\sigma), [X_{i1}^{(r)}, X_{j2}^{(r)}]] = 0.
\end{aligned}$$

For the relation (1.1.6)

$$[X_{ik}, Y_{jl}] = 0 \quad (i \neq j, k \neq l), \tag{1.1.6}$$

we need to show that

$$[S_{ik}^{(r)}, Y_{jl}^{(r)}] + [X_{ik}^{(r)}, T_{jl}^{(r)}] = 0. \tag{2.5.10}$$

If $(k, l) = (1, 2)$, we have

$$[S_{i1}^{(r)}, Y_{j2}^{(r)}] + [X_{i1}^{(r)}, T_{j2}^{(r)}] = \iota^{(r-1)}([S_{i1}^{(r-1)}, Y_{j2}^{(r-1)}] + [X_{i1}^{(r-1)}, T_{j2}^{(r-1)}]) = 0$$

by the assumption of induction (notice that $r > 2$). For general k and l, taking $\sigma \in \mathfrak{S}_r$ with $\sigma(k) = 1, \sigma(l) = 2$,

$$\begin{aligned}
&\sigma([S_{ik}^{(r)}, Y_{jl}^{(r)}] + [X_{ik}^{(r)}, T_{jl}^{(r)}]) \\
&= [S_{i1}^{(r)} + [a^{(r)}(\sigma), X_{i1}^{(r)}], Y_{j2}^{(r)}] + [X_{i1}^{(r)}, T_{j2}^{(r)} + [a^{(r)}(\sigma), Y_{j2}^{(r)}]] \\
&= [S_{i1}^{(r)}, Y_{j2}^{(r)}] + [X_{i1}^{(r)}, T_{j2}^{(r)}] + [a^{(r)}(\sigma), [X_{i1}^{(r)}, Y_{j2}^{(r)}]] = 0.
\end{aligned}$$

For the relation (1.1.7)

$$[X_{ik}, Y_{il}] = Z_{kl} \quad (k \neq l), \tag{1.1.7}$$

we need to show that

$$[S_{ik}^{(r)}, Y_{il}^{(r)}] + [X_{ik}^{(r)}, T_{il}^{(r)}] = [U_{kl}^{(r)}, Z_{kl}^{(r)}]. \tag{2.5.11}$$

If $(k, l) = (1, 2)$, we have

$$\begin{aligned}
&[S_{i1}^{(r)}, Y_{i2}^{(r)}] + [X_{i1}^{(r)}, T_{i2}^{(r)}] - [U_{12}^{(r)}, Z_{12}^{(r)}] \\
&= \iota^{(r-1)}([S_{i1}^{(r-1)}, Y_{i2}^{(r-1)}] + [X_{i1}^{(r-1)}, T_{i2}^{(r-1)}] - [U_{12}^{(r-1)}, Z_{12}^{(r-1)}]) \\
&= 0
\end{aligned}$$

by the assumption of induction (notice that $r > 2$). For general k and l, taking $\sigma \in \mathfrak{S}_r$ with $\sigma(k) = 1, \sigma(l) = 2$,

$$\begin{aligned}
&\sigma([S_{ik}^{(r)}, Y_{il}^{(r)}] + [X_{ik}^{(r)}, T_{il}^{(r)}] - [U_{kl}^{(r)}, Z_{kl}^{(r)}]) \\
=&[S_{i1}^{(r)} + [a^{(r)}(\sigma), X_{i1}^{(r)}], Y_{i2}^{(r)}] + [X_{i1}^{(r)}, T_{i2}^{(r)} + [a^{(r)}(\sigma), Y_{i2}^{(r)}]] \\
&- [U_{12}^{(r)} + a^{(r)}(\sigma), Z_{12}^{(r)}] \\
=&[S_{i1}^{(r)}, Y_{i2}^{(r)}] + [X_{i1}^{(r)}, T_{i2}^{(r)}] - [U_{12}^{(r)}, Z_{12}^{(r)}] + [a^{(r)}(\sigma), [X_{i1}^{(r)}, Y_{i2}^{(r)}] - Z_{12}^{(r)}] = 0.
\end{aligned}$$

Thus we have confirmed all the relations to examine.

3. Computation for genus one

In this section we shall present a result of actual computation in the case of genus one.

3.1. For the case of genus one we should consider a "superfluous" symmetry of Galois action as introduced in Nakamura-Takao [NTa] (2.4.2). Let E be an elliptic curve over $\mathbf{Q}$ with a $\mathbf{Q}$-rational point O, and $C = E \setminus \{O\}$. Then we can regard C as the configuration space $E^{(2)}$ of ordered two points on E divided by the translation action of E on it: $C \simeq E^{(2)}/E$. More generally, $C^{(r)}$ can be regarded as $E^{(r+1)}/E$ via

$$(3.1.1) \qquad \frac{E^{(r+1)}/E \simeq C^{(r)}}{\overline{(P_0, P_1, \ldots, P_r)} \leftrightsquigarrow (P_1 - P_0, \ldots, P_r - P_0).}$$

Since $E^{(r+1)}/E$ admits the action of $\mathfrak{S}_{r+1}$ by permutation, also $C^{(r)}$ has $\mathfrak{S}_{r+1}$-action via the above isomorphism extending the original $\mathfrak{S}_r$-action, where $\mathfrak{S}_r$ is identified with the stabilizer of the letter 0 in $\mathfrak{S}_{r+1} = \mathfrak{S}(\{0, 1, \ldots, r\})$. Since this action is defined over $\mathbf{Q}$, the action of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ commutes with this.

3.2. The graded Lie algebra $\mathcal{L}_1^{(r)}$ associated to the fundamental group of $\bar{C}^{(r)}$ is generated by $X_{1k}^{(r)}, Y_{1k}^{(r)}$ $(k = 1, \ldots, r)$ and $Z_{jk}^{(r)}$ $(j = 0, 1, \ldots, r; k = 1, \ldots, r)$. We put $X_{10}^{(r)} = -(\sum_{k=1}^{r} X_{1k}^{(r)}), Y_{10}^{(r)} = -(\sum_{k=1}^{r} Y_{1k}^{(r)})$ and $Z_{j0}^{(r)} = Z_{0j}^{(r)}$ $(j = 1, \ldots, r)$. Then the $\mathfrak{S}_{r+1}$-action corresponding to the one introduced above is described as $\sigma(X_{1k}^{(r)}) = X_{1,\sigma(k)}^{(r)}, \sigma(Y_{1k}^{(r)}) = Y_{1,\sigma(k)}^{(r)}$ and $\sigma(Z_{jk}^{(r)}) = Z_{\sigma(j),\sigma(k)}^{(r)}$ $(\sigma \in \mathfrak{S}_{r+1} = \mathfrak{S}(\{0, 1, \ldots, r\}))$. We denote by $\mathcal{D}_1^{(r)+}$ the subalgebra of $\mathcal{D}_1^{(r)}$ consisting of all W-normalized derivations which are $\mathfrak{S}_{r+1}$-invariant (not only $\mathfrak{S}_r$-invariant) modulo inner derivations. We want to know the structure of $\mathcal{D}_1^{(r)+}$.

PROPOSITION 3.3: *The Lie algebra $\mathcal{D}_1^{(r)+}$ consists of the even degree components of $\mathcal{D}_1^{(r)}$, that is,*

$$\mathrm{gr}^m \mathcal{D}_1^{(r)+} = \begin{cases} \mathrm{gr}^m \mathcal{D}_1^{(r)} & (m\text{: even}), \\ 0 & (m\text{: odd}). \end{cases}$$

Proof: When $r = 1$, $\mathfrak{S}_2 = \langle \sigma = (0\ 1) \rangle$ acts on $\mathcal{L}_1^{(1)}$ as $\sigma(X_{11}^{(1)}) = -X_{11}^{(1)}, \sigma(Y_{11}^{(1)}) = -Y_{11}^{(1)}$ and $\sigma(Z_{01}^{(1)}) = Z_{01}^{(1)}$. Hence σ acts as $(-1)^m$-multiplication on $\mathrm{gr}^m \mathcal{L}_1^{(1)}$, so on $\mathrm{gr}^m \mathcal{D}_1^{(r)}$.

When $r > 1$, since $\mathfrak{S}_{r+1}$ is generated by $\mathfrak{S}_r$ and $\sigma = (0\ 1)$, we must consider only the invariance by σ for an element of $\mathcal{D}_1^{(r)}$. The composite

$$\psi := \psi_2^{(2)} \circ \cdots \circ \psi_r^{(r)}\colon \mathrm{gr}^m \mathcal{D}_1^{(r)} \to \mathrm{gr}^m \mathcal{D}_1^{(1)}$$

commutes with the action of σ. Hence we have $\psi \circ \sigma = \sigma \circ \psi = (-1)^m \psi$. Since ψ is injective, the action of σ on $\mathrm{gr}^m \mathcal{D}_1^{(r)}$ is $(-1)^m$-multiplication. ∎

3.4. We shall consider the action of $\mathrm{GL}(2, \mathbf{Q})$ on $\mathcal{L}_1^{(r)}$ and $\mathcal{D}_1^{(r)+}$ tensored with $\mathbf{Q}$. Let $\mathcal{L}_{1,\mathbf{Q}}^{(r)} = \mathcal{L}_1^{(r)} \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathcal{D}_{1,\mathbf{Q}}^{(r)+} = \mathcal{D}_1^{(r)+} \otimes_{\mathbf{Z}} \mathbf{Q}$. The graded Lie algebra $\mathcal{L}_{1,\mathbf{Q}}^{(r)}$ has natural $\mathrm{GL}(2, \mathbf{Q})$-action as automorphisms ([NTs, NTaU, NTa, T]). This induces the action of $\mathrm{GL}(2, \mathbf{Q})$ on $\mathcal{D}_{1,\mathbf{Q}}^{(r)+}$ by conjugation. Hence its Lie algebra

$\mathfrak{gl}(2,\mathbf{Q})$ acts on $\mathcal{L}_{1,\mathbf{Q}}^{(r)}$ and on $\mathcal{D}_{1,\mathbf{Q}}^{(r)+}$ as derivations. Here we discuss this action of $\mathfrak{gl}(2,\mathbf{Q})$ on $\mathcal{L}_{1,\mathbf{Q}}^{(r)}$ and $\mathcal{D}_{1,\mathbf{Q}}^{(r)+}$ mainly based on [T].

3.5. Let $V = \mathbf{Q}^{\oplus 2} = \mathbf{Q}x \oplus \mathbf{Q}y$ be the standard representation of $\mathfrak{sl}(2,\mathbf{Q})$. To write it down explicitly, put

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They form a $\mathbf{Q}$-linear basis of $\mathfrak{sl}(2,\mathbf{Q})$ and h generates a Cartan subalgebra. When we identify a root α_n: $h \mapsto n$ with n, e (resp. f) is of weight 2 (resp. -2). The action of $\mathfrak{sl}(2,\mathbf{Q})$ on V is defined to be

$$(3.5.1) \qquad e\colon \begin{cases} x \mapsto 0, \\ y \mapsto x, \end{cases} \quad h\colon \begin{cases} x \mapsto x, \\ y \mapsto -y, \end{cases} \quad \text{and} \quad f\colon \begin{cases} x \mapsto y, \\ y \mapsto 0. \end{cases}$$

Then V is an irreducible $\mathfrak{sl}(2,\mathbf{Q})$-module of dimension 2 with a maximal vector x.

3.6. By applying the Witt formula together with $\mathfrak{gl}(2,\mathbf{Q})$-action, we obtain the irreducible decomposition of $\mathrm{gr}^m\,\mathcal{D}_{1,\mathbf{Q}}^{(1)+}$ as $\mathfrak{gl}(2,\mathbf{Q})$-modules. In the table we denote the symmetric tensor product $\mathrm{Sym}^n V$ by $[n]$, which is the unique $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbf{Q})$. On the $\mathfrak{sl}(2,\mathbf{Q})$-irreducible components $[n]$ in $\mathrm{gr}^m\,\mathcal{D}_{1,\mathbf{Q}}^{(r)+}$, $\mathfrak{gl}(2,\mathbf{Q})$ acts as $(\det^{\frac{m-n}{2}}) \odot [n]$. But we have no explicit formula on the irreducible decomposition of $\mathrm{gr}^m\,\mathcal{D}_{1,\mathbf{Q}}^{(2)+}$. In fact we do not know even dimensions of them as $\mathbf{Q}$-linear spaces in an explicit form. We, however, determined the $\mathfrak{gl}(2,\mathbf{Q})$-irreducible decomposition of $\mathrm{gr}^m\,\mathcal{D}_{1,\mathbf{Q}}^{(2)+}$ for $m \leq 12$ by checking the condition for derivations in $\mathrm{gr}^m\,\mathcal{D}_1^{(1)+}$ to be extended in $\mathrm{gr}^m\,\mathcal{D}_1^{(2)+}$ using computers, actually.

3.7. When $g = 1$, the first indices of X's and Y's are always 1. In this section we denote $X_{1j}^{(r)}, Y_{1j}^{(r)}$ simply by $X_j^{(r)}, Y_j^{(r)}$. The action of $\mathfrak{sl}(2,\mathbf{Q})$ on $\mathcal{L}_{1,\mathbf{Q}}^{(r)}$ is given by

$$(3.7.1) \quad e\colon \begin{cases} X_j^{(r)} \mapsto 0, \\ Y_j^{(r)} \mapsto X_j^{(r)}, \\ Z_{jk}^{(r)} \mapsto 0, \end{cases} \quad h\colon \begin{cases} X_j^{(r)} \mapsto X_j^{(r)}, \\ Y_j^{(r)} \mapsto -Y_j^{(r)}, \\ Z_{jk}^{(r)} \mapsto 0, \end{cases} \quad \text{and} \quad f\colon \begin{cases} X_j^{(r)} \mapsto Y_j^{(r)}, \\ Y_j^{(r)} \mapsto 0, \\ Z_{jk}^{(r)} \mapsto 0. \end{cases}$$

The action of $\mathfrak{sl}(2,\mathbf{Q})$ on $\mathcal{D}_{1,\mathbf{Q}}^{(r)}$ is given by $g\cdot D = g\circ D - D\circ g$ for $g \in \mathfrak{sl}(2,\mathbf{Q})$, $D \in \mathcal{D}_{1,\mathbf{Q}}^{(r)}$. These actions commute with $p_r^{(r)}, \psi_r^{(r)}$. Hence it suffices to consider which irreducible components of $\mathcal{D}_{1,\mathbf{Q}}^{(1)}$ are included in the image of $\mathcal{D}_{1,\mathbf{Q}}^{(r)}$. Moreover, it suffices to examine whether a maximal vector in each component of $\mathcal{D}^{(1)}_{1,\mathbf{Q}}$ can be extended to $\mathcal{D}^{(r)}_{1,\mathbf{Q}}$. Maximal vectors are characterized by the property to be killed by e.

3.8. Each derivation $D^{(1)} \in \mathcal{D}_1^{(1)+}$ is of the form

$$D^{(1)}\colon \begin{cases} X_1^{(1)} \mapsto S_1^{(1)} = \mathcal{S}(X_1^{(1)}, Y_1^{(1)}), \\ Y_1^{(1)} \mapsto T_1^{(1)} = \mathcal{T}(X_1^{(1)}, Y_1^{(1)}), \\ Z_{01}^{(1)} \mapsto 0 \end{cases} \tag{3.8.1}$$

with $[S_1^{(1)}, Y_1^{(1)}] + [X_1^{(1)}, T_1^{(1)}] = 0$. Here $\mathcal{S}$ and $\mathcal{T}$ are Lie polynomials of two variables.

PROPOSITION 3.9:

(1) *A derivation $D^{(1)}$ of the above form belongs to the image of $\psi_2^{(2)}\colon \mathcal{D}_1^{(2)+} \to \mathcal{D}_1^{(1)+}$ if and only if there exists an element $U \in \mathcal{L}_1^{(2)\circ}$ satisfying the following relations.*

$$U + \sigma(U) = 0 \quad (\sigma = (1\ 2)), \tag{3.9.1}$$
$$[Z_{01}^{(2)}, U] + [Z_{12}^{(2)}, \tau(U)] = 0 \quad (\tau = (0\ 2)), \tag{3.9.2}$$
$$[S_1^{(2)}, X_2^{(2)}] + [X_1^{(2)}, \sigma(S_1^{(2)})] = [X_2^{(2)}, [X_1^{(2)}, U]], \tag{3.9.3}$$
$$[T_1^{(2)}, Y_2^{(2)}] + [Y_1^{(2)}, \sigma(T_1^{(2)})] = [Y_2^{(2)}, [Y_1^{(2)}, U]], \tag{3.9.4}$$
$$[S_1^{(2)}, Y_2^{(2)}] + [X_1^{(2)}, \sigma(T_1^{(2)})] = [Z_{01}^{(2)}, U] + [X_1^{(2)}, [Y_2^{(2)}, U]]. \tag{3.9.5}$$

Here we put $S_1^{(2)} := i^{(1)}(S_1^{(1)}) = \mathcal{S}(X_1^{(2)}, Y_1^{(2)})$ and $T_1^{(2)} := i^{(1)}(T_1^{(1)}) = \mathcal{T}(X_1^{(2)}, Y_1^{(2)})$.

(2) *Assume that $D^{(1)}$ is a maxmal vector of weight w (i.e., $e\cdot D^{(1)} = 0, h\cdot D^{(1)} = wD^{(1)}$).*

   (a) *If $w = 0$, then we need to examine only (3.9.1), (3.9.2) and (3.9.5).*

   (b) *If $w \neq 0$, then U must be 0 and $D^{(1)}$ belongs to the image of $\psi_2^{(2)}$ if and only if*

$$[T_1^{(2)}, Y_2^{(2)}] + [Y_1^{(2)}, \sigma(T_1^{(2)})] = 0, \tag{3.9.6}$$
$$[S_1^{(2)}, Y_2^{(2)}] + [X_1^{(2)}, \sigma(T_1^{(2)})] = 0. \tag{3.9.7}$$

Proof: (1) Assume that $\psi_2^{(2)}(D^{(2)}) = D^{(1)}$. Then by Lemma 2.1, $D^{(2)}$ is of the following form:

$$\begin{aligned}
X_1^{(2)} &\longmapsto S_1^{(2)} = \mathcal{S}(X_1^{(2)}, Y_1^{(2)}),\\
Y_1^{(2)} &\longmapsto T_1^{(2)} = \mathcal{T}(X_1^{(2)}, Y_1^{(2)}),\\
X_2^{(2)} &\longmapsto S_2^{(2)} = \mathcal{S}(X_2^{(2)}, Y_2^{(2)}) + [U_{01}^{(2)}, X_2^{(2)}],\\
Y_2^{(2)} &\longmapsto T_2^{(2)} = \mathcal{T}(X_1^{(2)}, Y_1^{(2)}) + [U_{01}^{(2)}, Y_2^{(2)}],\\
Z_{12}^{(2)} &\longmapsto [U_{12}^{(2)}, Z_{12}^{(2)}],\\
Z_{01}^{(2)} &\longmapsto [U_{01}^{(2)}, Z_{01}^{(2)}],\\
Z_{02}^{(2)} &\longmapsto 0,
\end{aligned}$$

where $U_{01}^{(2)}, U_{12}^{(2)}$ belong to $\mathcal{L}_1^{(2)\circ}$ and satisfy the condition $U_{01}^{(2)} = U_{12}^{(2)} - \sigma(U_{12}^{(2)})$. Consider τ-invariance modulo inner derivations. Since $\tau = (0\ 2)$ fixes $W_1^{(2)} = Z_{01}^{(2)} + Z_{12}^{(2)}$ and $W_2^{(2)} = Z_{02}^{(2)}$, $\tau \circ D^{(2)} \circ \tau^{-1}$ is W-normalized. Hence $\tau \circ D^{(2)} \circ \tau^{-1} = D^{(2)}$. Since $\tau \circ D^{(2)} \circ \tau^{-1}(Z_{12}^{(2)}) = [\tau(U_{01}^{(2)}), Z_{12}^{(2)}]$, we have $U_{12}^{(2)} = \tau(U_{01}^{(2)})$. Therefore, if we put $U := U_{01}^{(2)}$, we have $\sigma(U) = \sigma(U_{12}^{(2)}) - U_{12}^{(2)} = -U$. The other relations are necessary for $D^{(2)}$ to satisfy the relations $[X_1^{(2)}, Y_1^{(2)}] + Z_{12}^{(2)} + Z_{01}^{(2)} = 0, [X_1^{(2)}, X_2^{(2)}] = 0, [Y_1^{(2)}, Y_2^{(2)}] = 0$ and $[X_1^{(2)}, Y_2^{(2)}] = Z_{12}^{(2)}$ in $\mathcal{L}_1^{(2)}$, respectively.

Conversely, assume that there exists $U \in \mathcal{L}_1^{(2)\circ}$ satisfying the above relations. Then, since $Z_{01}^{(2)} + Z_{12}^{(2)} + Z_{02}^{(2)}$ is central in $\mathcal{L}_1^{(2)\circ}$, we have

$$\begin{aligned}
[Z_{12}^{(2)}, U - \tau(U) + \sigma\tau(U)] &= [Z_{12}^{(2)}, U] + [Z_{01}^{(2)}, U] + \sigma([Z_{12}^{(2)}, \tau(U)])\\
&= [-Z_{02}^{(2)}, U] - \sigma([Z_{01}^{(2)}, U])\\
&= -[Z_{02}^{(2)}, U + \sigma(U)] = 0.
\end{aligned}$$

Hence it follows that $U = \tau(U) - \sigma\tau(U)$. Put $U_{01}^{(2)} := U, U_{12}^{(2)} := \tau(U)$ and define $D^{(2)}$ by the above assignment. The relations in the assumption assure that $D^{(2)}$ satisfies all of the defining relations of $\mathcal{L}_1^{(2)}$.

(2) Since $e \cdot D^{(1)} = 0$, we have $e \cdot S_1^{(1)} = 0, e \cdot T_1^{(1)} = S_1^{(1)}$. By operating e on both sides of (3.9.5), we obtain (3.9.3). Hence we can omit this.

(a) If $w = 0$, then $f \cdot D^{(1)} = 0$ and we have $f \cdot T_1^{(1)} = 0, f \cdot S_1^{(1)} = T_1^{(1)}$. By operating f on both sides of (3.9.5), we obtain (3.9.4). Hence we can omit this, too.

(b) Since h kills $\mathcal{L}_1^{(2)\circ}$, $wD^{(2)}(Z_{01}^{(2)}) = h \cdot D^{(2)}(Z_{01}^{(2)}) = 0$. Hence $U_{01}^{(2)}$ must be 0 if $w \neq 0$. The rest is obvious. ∎

Remark 3.10: By Lemma 1.15, $\mathcal{L}_1^{(2)\circ}$ can be identified with $\mathfrak{P}^{(4)} \simeq \mathcal{F}_2$ up to one-dimensional center $\langle Z_{01}^{(2)} + Z_{02}^{(2)} + Z_{12}^{(2)} \rangle$. In this sense we write $U = \mathcal{U}(Z_{02}^{(2)}, Z_{01}^{(2)})$.

Then (3.9.1) and (3.9.2) are written as

$$\mathcal{U}(Z_{02}^{(2)}, Z_{01}^{(2)}) + \mathcal{U}(Z_{01}^{(2)}, Z_{02}^{(2)}) = 0, \tag{3.10.1}$$

$$[Z_{01}^{(2)}, \mathcal{U}(Z_{02}^{(2)}, Z_{01}^{(2)})] + [Z_{12}^{(2)}, \mathcal{U}(Z_{02}^{(2)}, Z_{12}^{(2)})] = 0. \tag{3.10.2}$$

From this, it follows that $\mathcal{U}$ satisfies 2-cycle relation (A.5.2) and 3-cycle relations (A.5.3). Therefore, if we regard $D^{(2)}|_{\mathcal{L}_1^{(2)\circ}}$ as a derivation of $\mathfrak{P}^{(4)}$, it gives an element of $\mathcal{D}_0^{(4)}$.

3.11. By examining the condition of the above proposition, we determined which irreducible components of $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(1)}$ are in the image of $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(2)+}$ for $m \leq 12$.

Table 1. The multiplicities of the $\mathfrak{sl}(2,\mathbf{Q})$-irreducible decomposition of $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(r)+}$ $(r = 1, 2)$

m	$\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(1)+}$					
2	0[0]					
4	1[2]	+0[0]				
6	1[4]	+0[2]	+1[0]			
8	1[6]	+0[4]	+2[2]	+0[0]		
10	1[8]	+1[6]	+3[4]	+1[2]	+3[0]	
12	1[10]	+1[8]	+5[6]	+4[4]	+8[2]	+0[0]

m	$\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(2)+}$					
2	0[0]					
4	1[2]	+0[0]				
6	1[4]	+0[2]	+1[0]			
8	1[6]	+0[4]	+$\underline{1[2]}$	+0[0]		
10	1[8]	+1[6]	+$\underline{1[4]}$	+1[2]	+$\underline{1[0]}$	
12	1[10]	+1[8]	+$\underline{2[6]}$	+$\underline{2[4]}$	+$\underline{2[2]}$	+0[0]

The multiplicities of the underlined terms are truely less than those of $\mathcal{D}_{1,\mathbf{Q}}^{(1)}$. This shows the *non-surjectivity* of $\mathcal{D}_1^{(2)} \to \mathcal{D}_1^{(1)}$. This is the only known example of non-surjectivity of this kind of homomorphism other than $\mathcal{D}_0^{(5)} \to \mathcal{D}_0^{(4)}$, which is shown by Drinfeld and Ihara-Terada (see [I3]).

3.12. Let $\mathcal{G}_C$ be the Galois Lie algebra attached to $\pi_1^{\text{pro}-l}(\bar{C})$. Then $\mathcal{G}_C \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ is included in $\mathcal{D}_{1,\mathbf{Q}}^{(r)+} \otimes_{\mathbf{Q}} \mathbf{Q}_l$. If E has no complex multiplication, the (abelian) l-adic representation $\rho_E\colon G_{\mathbf{Q}} \to \mathrm{GL}(2, \mathbf{Z}_l)$ has an open image (Serre [Se]). The image $\rho_E(G_{\mathbf{Q}})$ acts on $\mathcal{G}_C$ by conjugation. This induces the action of $\mathfrak{gl}(2, \mathbf{Z}_l)$ on $\mathcal{G}_C$ as derivations. When both are tensored with $\mathbf{Q}_l$, the action of $\mathfrak{gl}(2, \mathbf{Z}_l)$ on $\mathcal{G}_C$ and the action of $\mathfrak{gl}(2, \mathbf{Q})$ on $\mathcal{D}_{1,\mathbf{Q}}^{(r)+}$ are equivariant, that is, the inclusion $\mathcal{G}_C \otimes_{\mathbf{Z}_l} \mathbf{Q}_l \hookrightarrow \mathcal{D}_{1,\mathbf{Q}}^{(r)+} \otimes_{\mathbf{Q}} \mathbf{Q}_l$ is $\mathfrak{gl}(2, \mathbf{Q}_l)$-equivariant. Hence the determination of the Galois image in $\mathcal{D}_{1,\mathbf{Q}}^{(r)+} \otimes_{\mathbf{Q}} \mathbf{Q}_l$ means the determination of $\mathfrak{gl}(2, \mathbf{Q}_l)$-irreducible components containing the Galois images.

We know non-trivial elements in $\mathcal{G}_C$ coming from two different origins.

3.13. By the theory of universal monodromy representation, we have a canonical surjection

$$\psi_C\colon \mathcal{G}_C \twoheadrightarrow \mathcal{G}_{1,1} = \mathcal{G}_{0,3} = \mathcal{G}_{\mathbf{P}^1 \setminus \{0,1,\infty\}} \tag{3.13.1}$$

(see also the next section). Oda [O] showed that $\mathrm{SL}(2, \mathbf{Z}_l)$ acts trivially on the irreducible components which survives the above surjection. Since $\mathcal{G}_{0,3} \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ has one generator in each odd degree greater than or equal to 3, $\mathcal{G}_C$ has one [0]-component in each degree $4m+2$ $(m > 1)$. In degree 6 and 10, since the multiplicity of [0]-component in $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(2)+}$ is one (and since the images of the generators by the derivation in this component are determined by actual computation), this property characterizes the Galois image component coming from this nature in $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(1)+}$. We denote by D_m^0 one of the non-vanishing elements under ψ_C.

3.14. By considering the action of $G_{\mathbf{Q}}$ on the meta-abelian quotient $\pi_1^{\mathrm{pro}-l}(\bar{C})/\pi_1^{\mathrm{pro}-l}(\bar{C})''$, Nakamura [N1] showed that for a generic elliptic curve E there are non-trivial Galois images in the highest weight component $[m-2]$ in $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(r)+}$ for each even m greater than or equal to 4. Because the multiplicity of $[m-2]$ in $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(1)+}$ is one for all even $m \geq 4$, the Galois image component coming from this nature is characterized by this property in $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(1)+}$. A maximal vector $D_m = D_m^{(1)}$ is given by

$$D_m^{(1)}\colon \begin{cases} X_1^{(1)} \longmapsto (\operatorname{Ad} X_1^{(1)})^m Y_1^{(1)}, \\ Y_1^{(1)} \longmapsto \sum\limits_{r=0}^{\frac{m}{2}-1} (-1)^r [(\operatorname{Ad} X_1^{(1)})^r Y_1^{(1)}, (\operatorname{Ad} X_1^{(1)})^{m-1-r} Y_1^{(1)}], \\ Z_{01}^{(1)} \longmapsto 0. \end{cases} \tag{3.14.1}$$

PROPOSITION 3.15:

(1) *For a generic elliptic curve E, we have*

$$\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(2)+} \otimes_{\mathbf{Q}} \mathbf{Q}_l \simeq \mathrm{gr}^m \mathcal{G}_C \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$$

when $m \leq 12$.

(2) *For any* $r \geq 3$,

$$\mathrm{gr}^m \mathcal{D}_1^{(2)+} = \mathrm{gr}^m \mathcal{D}_1^{(r)+}$$

when $m \leq 12$.

Proof: It suffices to show that all the irreducible components of $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(2)+}$ are generated by two kinds of Galois image components mentioned above. In Table 2 we list maximal vectors of all the irreducible components of $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(2)+}$ ($m \leq 12$). By actual computation we can confirm that all of them are non-zero and linearly independent. ∎

Table 2. Maximal vectors of $\mathrm{gr}^m \mathcal{D}_{1,\mathbf{Q}}^{(2)+}$

Degree	Weight	A maximal vector
$m = 4$	$w = 2$	D_4
6	4	D_6
	0	D_6^0
8	6	D_8
	2	$[D_4, f \cdot D_4]$
10	8	D_8
	6	$[D_4, D_6]$
	4	$[D_4, f \cdot D_6] - 2[f \cdot D_4, D_6]$
	2	$[D_4, f \cdot f \cdot D_6] - 3[f \cdot D_4, f \cdot D_6] + 6[f \cdot f \cdot D_4, D_6]$
	0	D_{10}^0
12	10	D_{12}
	8	$[D_4, D_8]$
	6	$[D_4, f \cdot D_8] - 3[f \cdot D_4, D_8]$
		$[D_6, f \cdot D_6]$
	4	$[D_4, f \cdot f \cdot D_8] - 5[f \cdot D_4, f \cdot D_8] + 15[f \cdot f \cdot D_4, D_8]$
		$[D_4, [D_4, f \cdot D_4]]$
	2	$2[D_6, f \cdot f \cdot f \cdot D_6] - 3[f \cdot D_6, f \cdot f \cdot D_6]$
		$[D_4, [D_4, f \cdot f \cdot D_4]] - [f \cdot D_4, [D_4, f \cdot D_4]]$

Note that our stability theorem assures that $\mathcal{D}_1^{(3)+} = \mathcal{D}_1^{(r)+}$ for any $r \geq 4$.

4. Further discussion

4.1. For any curve C of type (g, n), the Galois Lie algebra $\mathcal{G}_C$ has a quotient $\mathcal{G}_{g,n}$ which depends only on (g, n), owing to the theory of universal monodromy representations (Oda [O]). Moreover, $\mathcal{G}_{g,n}$ does not depend even on (g,n) (Oda's conjecture, proved by [N2, NTaU, IN, M]). Hence we always have a natural surjection

$$(4.1.1) \qquad \psi_C\colon \mathcal{G}_C \longrightarrow \mathcal{G}_{g,n} = \mathcal{G}_{0,3} = \mathcal{G}_{\mathbf{P}^1\setminus\{0,1,\infty\}}.$$

Define a subalgebra $\mathcal{D}_g^{(r)0}$ of $\mathcal{D}_g^{(r)}$ by

$$(4.1.2) \qquad \mathcal{D}_g^{(r)0} = \{D \in \mathcal{D}_g^{(r)} \mid D|_{\mathcal{L}_g^{(r)\circ}} = 0\}.$$

Since $D \in \mathcal{D}_g^{(r)}$ stabilizes $\mathcal{L}_g^{(r)\circ}$, $\mathcal{D}_g^{(r)0}$ is a Lie ideal of $\mathcal{D}_g^{(r)}$. Consider a surjection $\mathcal{D}_g^{(r)} \to \mathcal{D}_g^{(r)}/\mathcal{D}_g^{(r)0}$ obtained by restricting a derivation on $\mathcal{L}_g^{(r)\circ}$. Matsumoto's result [M] shows that the composite $\mathcal{G}_C \hookrightarrow \mathcal{D}_g^{(r)} \to \mathcal{D}_g^{(r)}/\mathcal{D}_g^{(r)0}$ factors through ψ_C. Compiling these facts, we obtain the following diagram:

$$(4.1.3) \qquad \begin{tikzcd}
\mathcal{G}_C \arrow[r, hook] \arrow[d] & \mathcal{D}_g^{(3)} \arrow[r, hook] \arrow[d] & \mathcal{D}_g^{(2)} \arrow[r, hook] \arrow[d] & \mathcal{D}_g^{(1)} \arrow[d] \\
\mathcal{G}_{g,1} \arrow[r, hook] \arrow[d, equal] & \mathcal{D}_g^{(3)}/\mathcal{D}_g^{(3)0} \arrow[r, hook] \arrow[d] & \mathcal{D}_g^{(2)}/\mathcal{D}_g^{(2)0} \arrow[r] \arrow[dr] & 0 \\
\mathcal{G}_{0,3} \arrow[r, hook] & \mathcal{D}_0^{(5)} \arrow[r, hook] & \mathcal{D}_0^{(4)} \arrow[r, hook] & \mathrm{Out}^\flat\, \mathcal{P}^{(4)}
\end{tikzcd}$$

Here, the homomorphisms $\mathcal{D}_g^{(r)}/\mathcal{D}_g^{(r)0} \to \mathrm{Out}^\flat\, \mathcal{P}^{(r+2)}$ are the injections obtained by the identification in Lemma 1.15, whose images are included in $\mathcal{D}_0^{(r+2)}$ when $r \geq 3$ by "automatic symmetry" (Proposition 1.13). When $g = 1$, the image of $\mathcal{D}_1^{(2)+}$ is included in $\mathcal{D}_0^{(4)}$ because of $\mathfrak{S}_{2+1}$-symmetricity.

Concerning the above diagram (4.1.3), we want to pose the following questions 4.2 and 4.3

4.2. IS $\mathcal{D}_g^{(3)} \to \mathcal{D}_g^{(2)}$ SURJECTIVE? This inquires whether $r = 3$ is best possible or not in our theorem. As seen in the last section, $\mathcal{D}_1^{(2)+} \to \mathcal{D}_1^{(1)+}$ is *not* surjective, and $\mathrm{gr}^m\, \mathcal{D}_1^{(3)+} \to \mathrm{gr}^m\, \mathcal{D}_1^{(2)+}$ is surjective for $m \leq 12$ because the composite $\mathrm{gr}^m\, \mathcal{G}_C \otimes_{\mathbf{Z}_l} \mathbf{Q}_l \to \mathrm{gr}^m\, \mathcal{D}_{1,\mathbf{Q}}^{(3)+} \otimes_{\mathbf{Q}} \mathbf{Q}_l \to \mathrm{gr}^m\, \mathcal{D}_{1,\mathbf{Q}}^{(2)+} \otimes_{\mathbf{Q}} \mathbf{Q}_l$ is surjective. The actual computation in the component of degree 14 seems very interesting and important for the following reasons. Suppose that $\mathcal{D}_g^{(3)} \to \mathcal{D}_g^{(2)}$ would be surjective. If $D_0 \in \mathcal{D}_0^{(4)}$ could be lifted to an element $D^{(2)} \in \mathcal{D}_g^{(2)}$, then D_0 must be lifted to an element in $\mathcal{D}_0^{(5)}$, which is obtained by the image of the element in $\mathcal{D}_g^{(3)}$ extending $D^{(2)}$. This means that the 5-cycle relation (A.8.1), which distinguishes $\mathcal{D}_0^{(5)}$ from $\mathcal{D}_0^{(4)}$, could be recovered from the extendability from $\mathcal{D}_0^{(4)}$ to $\mathcal{D}_g^{(2)}$. This computation is, however, very hard to carry out since the growth of degree seems to cause a serious increase of complexity of computation (maybe of exponential growth).

4.3. IS $\mathcal{D}_g^{(3)} \to \mathcal{D}_0^{(5)}$ SURJECTIVE? Since $\mathcal{G}_{0,3} \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ is included in the image of $\mathcal{D}_g^{(3)} \otimes_{\mathbf{Q}} \mathbf{Q}_l$ in $\mathcal{D}_0^{(5)} \otimes_{\mathbf{Z}} \mathbf{Q}_l$, this question is related to the problem whether $\mathcal{G}_{0,3} \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ coincides with $\mathcal{D}_0^{(5)} \otimes_{\mathbf{Z}} \mathbf{Q}_l$ or not, which is regarded as a graded Lie algebra version of the open problem whether $G_{\mathbf{Q}}$ coincides with the Grothendieck-Teichmüller group GT or not. If we suppose $\mathcal{G}_{0,3} \otimes_{\mathbf{Z}_l} \mathbf{Q}_l = \mathcal{D}_0^{(5)} \otimes_{\mathbf{Z}} \mathbf{Q}_l$, it implies that $\mathcal{D}_g^{(3)} \to \mathcal{D}_0^{(5)}$ is surjective. But this surjectivity does not seem easy as a problem on graded Lie algebras. On the other hand, if $\mathcal{G}_{0,3} \otimes_{\mathbf{Z}_l} \mathbf{Q}_l \subsetneq \mathcal{D}_0^{(5)} \otimes_{\mathbf{Z}} \mathbf{Q}_l$ would hold, then we want to ask whether the obstruction comes from the extendability from $\mathcal{D}_0^{(5)}$ to $\mathcal{D}_g^{(3)}$.

Appendix A. Review of the case of genus zero

Here we shall recall Ihara's result [I3] on the stability properties of derivation algebras associated with a sphere (genus zero) braid group on n strings. We also add some new remarks.

A.1. For $n \geq 4$, let $\mathfrak{P}^{(n)}$ be the graded Lie algebra over $\mathbf{Z}$ defined by the following generators and relations:

generators: $x_{ij}^{(n)} = x_{ij}$ $(i, j = 1, \ldots, n)$;

relations:

$$x_{ii} = 0, \quad x_{ij} = x_{ji} \quad (1 \leq i, j \leq n), \tag{A.1.1}$$

$$\sum_{j=1}^{n} x_{ij} = 0 \quad (1 \leq i \leq n), \tag{A.1.2}$$

$$[x_{ij}, x_{kl}] = 0 \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset. \tag{A.1.3}$$

We denote the homogeneous component of $\mathfrak{P}^{(n)}$ of degree m by $\operatorname{gr}^m \mathfrak{P}^{(n)}$. The symmetric group $\mathfrak{S}_n = \mathfrak{S}(\{1, \ldots, n\})$ acts on $\mathfrak{P}^{(n)}$ by $\sigma(x_{ij}) = x_{\sigma(i)\sigma(j)}$.

Remark A.2: In [I3], Ihara defined $\mathfrak{P}^{(n)}$ as a graded Lie algebra over $\mathbf{Q}$. But, as is considered in [I4], it can be formulated over $\mathbf{Z}$. Moreover, he discussed the relationship between a certain congruence property of stable derivations modulo irregular primes and arithmetic of cyclotomic fields.

A.3. When $n = 4$, the defining relations imply that $x := x_{12} = x_{34}, y := x_{13} = x_{24}, z := x_{14} = x_{23}$ and $x + y + z = 0$. Hence $\mathfrak{P}^{(4)}$ is isomorphic to the free Lie algebra $\mathcal{F}_2$ of rank two and the action of $\mathfrak{S}_4$ factors through its quotient

$\mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_4/V_4 \simeq \mathfrak{S}_3$ (where $V_4 = \langle (1\ 2)(3\ 4), (1\ 3)(2\ 4)\rangle$ is the Klein's four group included in $\mathfrak{S}_4$) as substitutions of x, y, z.

A.4. For $n \geq 5$, we have the "forgetting the n-th string" homomorphism $\mathrm{pr}_n : \mathfrak{P}^{(n)} \to \mathfrak{P}^{(n-1)}$ defined by $x_{ij}^{(n)} \mapsto x_{ij}^{(n-1)}$ for $1 \leq i, j \leq n-1$ and $x_{in}^{(n)} \mapsto 0$ for $1 \leq i \leq n$.

A.5. A derivation D of $\mathfrak{P}^{(n)}$ is said to be **special** (or **braid-like**) if for each i, j $(1 \leq i, j \leq n)$ there exists some $t_{ij} \in \mathfrak{P}^{(n)}$ such that $D(x_{ij}) = [t_{ij}, x_{ij}]$. The set of all special derivations $\mathrm{Der}^\flat \mathfrak{P}^{(n)}$ of $\mathfrak{P}^{(n)}$ forms a graded Lie algebra. All inner derivations are special and all of them form a homogeneous Lie ideal $\mathrm{Int}\, \mathfrak{P}^{(n)}$. The quotient $\mathrm{Out}^\flat \mathfrak{P}^{(n)} := \mathrm{Der}^\flat \mathfrak{P}^{(n)} / \mathrm{Int}\, \mathfrak{P}^{(n)}$ is called the algebra of special outer derivations. It admits a natural $\mathfrak{S}_n$-action. We denote by $\mathcal{D}_0^{(n)}$ the $\mathfrak{S}_n$-fixed subalgebra of $\mathrm{Out}^\flat \mathfrak{P}^{(n)}$. Each class of $\mathcal{D}_0^{(n)}$ is represented by a special derivation which is $\mathfrak{S}_n$-invariant modulo inner derivations. We prefer considering suitable representatives to considering classes of derivations modulo inner derivations. To choose such representatives, we introduce a system of elements of $\mathfrak{P}^{(n)}$ for normalization. For $i = 2, \ldots, n-1$, put $y_i = y_i^{(n)} := \sum_{j=1}^{i-1} x_{ij}^{(n)} = -\sum_{j=i+1}^{n} x_{ij}^{(n)}$. A special derivation D of $\mathfrak{P}^{(n)}$ is said to be *y-normalized* if $D(y_i) = 0$ for $i = 2, \ldots, n-1$. Then each class of $\mathrm{Out}^\flat \mathfrak{P}^{(n)}$ of degree greater than one is represented by a unique y-normalized special derivation, that is, there exists a unique element $f = f(x, y) \in \mathfrak{P}^{(4)} \simeq \mathcal{F}_2$, such that it is represented by the derivation $D_f = D_f^{(4)}$ defined to be

$$D_f(x) = 0, \quad D_f(y) = [y, f(x, y)] \tag{A.5.1}$$

([I3] Proposition 1). Conversely, D_f defined as above represents a class in $\mathcal{D}_0^{(4)}$ if and only if f satisfies

$$f(x, y) + f(y, x) = 0, \tag{A.5.2}$$

$$[y, f(x, y)] + [z, f(x, z)] = 0. \tag{A.5.3}$$

Moreover, (A.5.2) and (A.5.3) imply

$$f(x, y) + f(y, z) + f(z, x) = 0 \tag{A.5.4}$$

([I3] Proposition 2).

A.6. Since a special derivation stabilizes the kernel of pr_n, pr_n induces a homomorphism $\tilde{\psi}^{(n)}$: $\mathrm{Der}^\flat \mathfrak{P}^{(n)} \to \mathrm{Der}^\flat \mathfrak{P}^{(n-1)}$ and $\psi^{(n)}$: $\mathrm{Out}^\flat \mathfrak{P}^{(n)} \to \mathrm{Out}^\flat \mathfrak{P}^{(n-1)}$. If we identify $\mathfrak{S}_{n-1}$ with the stabilizer of the index n in $\mathfrak{S}_n$, these morphisms are $\mathfrak{S}_{n-1}$-equivariant. Hence, by restricting $\psi^{(n)}$ on the $\mathfrak{S}_n$-fixed part $\mathcal{D}_0^{(n)}$, we have

$$\psi^{(n)}\colon \mathcal{D}_0^{(n)} \longrightarrow \mathcal{D}_0^{(n-1)}. \tag{A.6.1}$$

THEOREM A.7 (Ihara [I2]): *For $n \geq 5$ (that is, for all cases)*

$$\psi^{(n)}\colon \operatorname{Out}^{\flat}\mathfrak{P}^{(n)} \longrightarrow \operatorname{Out}^{\flat}\mathfrak{P}^{(n-1)}$$

is injective.

The following theorem is the main result of Ihara [I3], which is used in the proof of the main result of this paper at the most crucial point.

THEOREM A.8 (Ihara [I3]):

(1) *The class $\{D_f^{(4)}\}$ belongs to the image of $\psi^{(5)}\colon \mathcal{D}_0^{(5)} \to \mathcal{D}_0^{(4)}$ if and only if f satisfies the following 5-cycle relation in $\mathfrak{P}^{(5)}$,*

(A.8.1)

$$f(x_{12},x_{23}) + f(x_{34},x_{45}) + f(x_{51},x_{12}) + f(x_{23},x_{34}) + f(x_{45},x_{51}) = 0.$$

(2) *For $n \geq 6$ (that is, except for the first stage),*

$$\psi^{(n)}\colon \mathcal{D}_0^{(n)} \longrightarrow \mathcal{D}_0^{(n-1)}$$

is surjective (hence bijective).

Remark A.9: Here we shall make some remarks. The first one is a relation between the $\mathfrak{S}_n$-symmetricity of outer special derivations and the injectivity of $\psi^{(n)}$. Theorem A.7 asserts that $\psi^{(n)}$ is injective even not restricted on the $\mathfrak{S}_n$-fixed part. This leads us to the automatic symmetricity of special outer derivations.

PROPOSITION A.10:

(1) *Identify $\mathfrak{S}_{n-1}$ with the stabilizer of the index n in $\mathfrak{S}_n$. For $\{D\} \in \operatorname{Out}^{\flat}\mathfrak{P}^{(n)}$ and $\sigma \in \mathfrak{S}_{n-1}$, if $\psi^{(n)}(\{D\})$ is fixed by σ, then $\{D\}$ is fixed by $\sigma \in \mathfrak{S}_{n-1} \subset \mathfrak{S}_n$.*

(2) *The alternating group $\mathfrak{A}_5$ of degree 5 acts on $\operatorname{Out}^{\flat}\mathfrak{P}^{(5)}$ trivially.*

(3) *If $n \geq 6$, $\mathfrak{S}_n$ acts on $\operatorname{Out}^{\flat}\mathfrak{P}^{(n)}$ trivially. Hence $\mathcal{D}_0^{(n)} = \operatorname{Out}^{\flat}\mathfrak{P}^{(n)}$.*

Proof: (1) Since $\psi^{(n)}$ is $\mathfrak{S}_{n-1}$-equivariant under the above identification, if $\sigma(\psi^{(n)}(\{D\})) = \psi^{(n)}(\{D\})$ for $\sigma \in \mathfrak{S}_{r-1}$, then we have

$$\psi^{(n)}(\{D\}) = \sigma(\psi^{(n)}(\{D\})) = \psi^{(n)}(\sigma(\{D\})).$$

Hence $\{D\} = \sigma(\{D\})$ follows from the injectivity of $\psi^{(n)}$.

(2) The Klein's four group V_4 acts trivially on $\mathcal{P}^{(4)}$, so on $\mathrm{Out}^\flat\,\mathfrak{P}^{(4)}$. Since $\mathfrak{A}_5$ is normally generated by V_4 in $\mathfrak{S}_5$, it acts on $\mathrm{Out}^\flat\,\mathfrak{P}^{(5)}$ trivially by (1).

(3) It suffices to show that a transposition $\sigma = (n-1\ n)$ acts on $\mathrm{Out}^\flat\,\mathfrak{P}^{(n)}$ trivially because it normally generates $\mathfrak{S}_n$ in it. It follows from the injectivity of $\psi^{(n)}$ and $\psi^{(n-1)}$ and that $\psi^{(n-1)}\circ\psi^{(n)} = \psi^{(n-1)}\circ\psi^{(n)}\circ\sigma$.

Remark A.11: Drinfel'd [Dr] proved that 2-, 3-, and 5-cycle relations (A.5.2), (A.5.4) and (A.8.1) imply (A.5.3). We shall remark that if the characteristic of the coefficient ring is not zero, this implication fails. For example, $f = [x,[x,[x,y]]] -[y,[x,[x,y]]] + [y,[y,[x,y]]]$ satisfies (A.5.2), (A.5.4) and (A.8.1) modulo 5, but does not satisfy (A.5.3) modulo 5.

Remark A.12: Although it is not introduced explicitly in Ihara [I3], it seems useful for understanding the contents of the paper to consider the "duplicating the $(n-1)$-th string" homomorphism

$$\text{(A.12.1)}\qquad \begin{aligned} \iota_{n-1}\colon \mathfrak{P}^{(n-1)} &\longrightarrow C_{\mathfrak{P}^{(n)}}(x_{n-1,n})/\langle x_{n-1,n}\rangle \\ x_{ij}^{(n-1)} &\longmapsto x_{ij}^{(n)} \bmod \langle x_{n-1,n}\rangle \quad (1\le i,j\le n-1). \end{aligned}$$

For $m>1$ it gives a injective homomorphism $\mathrm{gr}^m\,\mathfrak{P}^{(n-1)} \to \mathrm{gr}^m\,\mathfrak{P}^{(n)}$, which is a section of $\mathrm{pr}_n : \mathrm{gr}^m\,\mathfrak{P}^{(n)} \to \mathrm{gr}^m\,\mathfrak{P}^{(n-1)}$ and is characterized as the unique section whose image is contained in the centralizer $C_{\mathfrak{P}^{(n)}}(x_{n-1,n})$ of $x_{n-1,n}$ in $\mathfrak{P}^{(n)}$.

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